A GENERAL COURSE OF
PURE MATHEMATICS

FROM INDICES TO
SOLID ANALYTICAL GEOMETRY

BY

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PREFACE

THIS book is the result of an attempt to bring within two covers a wide region of pure mathematics. Knowledge is assumed of that part of mathematics usually required for matriculation, namely algebra to simultaneous quadratic equations and the substance of the first four books of Euclid, together with a very slight acquaintance with graphic algebra, mensuration, and solid geometry. From this stage the work is carried forward in algebra to the logarithmic series; in co-ordinate geometry to the nature of the general conicoid; in trigonometry to the use of Euler's expressions for the sine and cosine, with a careful treatment of imaginary quantities; in calculus to definite integration and to the maxima of a function of n independent variables; together with the pure geometry which is necessary for the other subjects. It has been the intention to include the bulk of the results obtained in pure mathematics which admit of rigid proof of a fairly easy character, and are needed by those who use pure mathematics as an instrument in mechanics, engineering, physics, chemistry, and economics. For this purpose a very great deal that is ordinarily contained in text-books has been thrown aside, and only those theorems and formulae which are of direct practical application or which are necessary to lead to others of direct practical application are retained.

It has also been the intention to give exact definitions and strict proofs, of a more careful nature than those found in many of the more diffuse and elementary books; only two difficulties have been intentionally glozed over, viz. the nature of continuity and the nature of irrationals. Continuity is best understood after a considerable knowledge of processes and of functions is obtained; experience must precede definition; in this book the appeal is to graphic experience, and the solution of equations and the determination of maxima and minima are treated to some extent empirically. The modern theory of irrationals is evidently unnecessary except for the mathematical specialist. Nevertheless the great majority of the functions which are tabulated in mathematical tables, as being of specially practical use, are irrational for nearly all values of the variables, and therefore the method of the passage from rational to irrational, from commensurable to incommensurable, must be faced. In this book two methods are used: one, in trigonometry and analytical geometry, uses the assumption that all numerical quantities concerned can be measured by the distance between two points on a straight line; in the other, in logarithms, limits, &c., irrationals are always approximated to by the use of neighbouring rationals.

It is very commonly the case in text-books that powers are interpreted, and the exponential theorem proved, on the assumption that the index is rational, and that then they are forthwith used for logarithms which are irrational. In this, and several other cases, a quite unpretentious attempt has been made to restore to elementary mathematics part of the exactness which writers have sacrificed in the desire to make the subject easy and attractive. In particular the theory of imaginary quantities has been recast. I have never been able to understand, nor to believe in the logical justification of, the accepted treatment of imaginaries; in effect it generally begins 'let ι be a quantity such that $\iota \times \iota = -1$', and continues 'multiplication by ι can be represented by rotation of a quantity through a right angle'; but the latter process is only illustrative and does not make a definition, and the former involves two conceptions in one definition, viz. ι and multiplication, and there is nothing in any previous use of $\times$ to show how it is to be applied when the multiplier is not real. Further, the meaning of $+$ in the expression $x+y\iota$ is never defined.

I have therefore based the work quite differently, namely on the use of an operator which when repeated reverses the sign of the quantity operated on, and have followed with definitions of and rules for the use of the symbols used. So far as I know this method has not hitherto been used in just this way, though all the ideas involved are quite familiar, and have been since the time of Hamilton's invention of quaternions. It appears that writers of elementary text-books have been content to follow each other in Cauchy's steps, and that mathematical pioneers have not had occasion to level out this particular field.

Of the overgrowth of algebra, trigonometry, and 'conic sections' that has been cut away, much is purely traditional (as Euclid's treatment of proportion), much is the invention of the compiler of the cramming text-book, much of the pedagogue anxious to occupy his boys' time. Here Ratio, Proportion, Variation and Progressions are reduced to a very small bulk; Indices are only developed as leading to logarithms (till we come to series); Permutations and Combinations are only wanted for the Binomial series, except for the specialist in the Theory of Probability; trigonometrical identities are reduced to a utilitarian minimum.

There seems no reason why the best years of a scholar's life should be devoted to the Conic Section, treated geometrically and analytically in Cartesian co-ordinates. Geometrical Conics is a barren field, till it is impregnated by modern geometry. The controversy between Descartes and his contemporaries has still a vicious influence in the separation of geometry from analysis and the duplication of proofs. The sixty pages of Section VI will be found to contain the most familiar elementary analytical results[1] as well as many of the more purely geometrical properties, together with what is much more important—a complete analysis of the equation of the second degree. The time thus saved allows an excursion in Section IX into three dimensions, where the plane analysis can be reviewed from without, and where many methods and results of great importance to the physicist and to other

[1] Except co-axal circles, which belong to elementary geometry, and the equations of two straight lines and their bisectors, which are only a special case.

applied mathematicians are brought down from the heaven of advanced, bulky, and expensive treatises where they have mostly resided.

The book is intended to form a protest against the multiplication and separation of mathematical subjects, and against the enormous waste of time involved in the course now generally followed in the upper forms of Schools and the pass work of Universities, that results in most students, who are not mathematical specialists but want the subject for further use, never arriving at any general knowledge of the methods or theorems they need. Otherwise there has been no definite aim at originality. The order of treatment has been recast in detail, always with the idea of reaching important and advanced results by the simplest and shortest legitimate route, but most of the proofs are those given in one known treatment or another. The first six sections have been developed, so far as choice of order and proofs is concerned, in lectures at University College, Reading, and the London School of Economics. In many cases I have subsequently found in new text-books methods and proofs I had already in use, showing, I hope, that the treatment is consonant with modern ideas of teaching. I trust that I have not unintentionally used proofs without acknowledgement which other recent writers have discovered and whose origin I have forgotten.

The first three sections are cut down to their minimum, as it is expected that readers will in fact have some preliminary acquaintance with the subject-matter and only need a systematic revision. Logarithmic solution of triangles is omitted, since it is unnecessary for the sequel, and land-surveyors and others who have practical work to do will in any case need a separate book. Section VII does not pretend to be more than an introduction to the Calculus, though (as modern writers have shown) many of the most useful results can be obtained in a very brief argument. Mathematical physicists and many other scientists cannot do without a much more elaborate and general treatment. The short and easy subject of Spherical Trigonometry is omitted as being mainly of specialist use and readily accessible.

It is hoped that the book will supply a general and accurate view of that intermediate region of pure mathematics on which are based many of the results used in other subjects. Specialists in any branch will need in addition a treatise relating to the subject-matter of their profession. It may also afford a convenient book of reference to teachers, who now and again want proofs and methods outside their routine. It should form a useful means of revision of the principal body of theory needed in the pass examinations of modern Universities. It is not intended for the immature or for those whose mind is not naturally mathematical, and it will not in itself afford sufficient experience in the handling of mathematical expressions—that should be obtained in connexion with the particular branch which may be specially needed.

My thanks are due to my colleagues Miss L. Ashcroft and Mr. H. Knapman, who read critically the original manuscript; to Mr. G. W. Palmer (of Christ's Hospital), who has made many valuable suggestions at various stages of the work; and especially to my colleague Mr. J. P. Clatworthy, who has worked critically point by point through the manuscript and proof at every stage and has drawn the diagrams.

A. L. BOWLEY.

University College, Reading,
March, 1913.

Note.—The reader can proceed to the great part of Section VI immediately after Section IV, and thence to Section IX. Also Section VIII can be taken immediately after Section V, and the great part of Section VII can be followed without reference to Section VI.

Logarithms to Base e.

Consider the equation $y = e^x$. e^x has only been assigned a meaning when x is commensurable. e is incommensurable. e^x and y cannot be assumed to be commensurable, but approximate values can be found as follows.

Suppose a network to be ruled in the figure representing commensurable values on the scale of x and y. The lines may be supposed to be as near as we please. Let one of these lines parallel to OX meet the curve representing e^x at P_1, and let P_1 fall between two adjacent vertical lines, $m_1 p_1$, $m_2 p_2$, meeting the curve at p_1, p_2. Let $y_1 = M_1 P_1$, $x_1 = Om_1$, $\delta = m_1 m_2$, x_1 and δ being positive and commensurable.

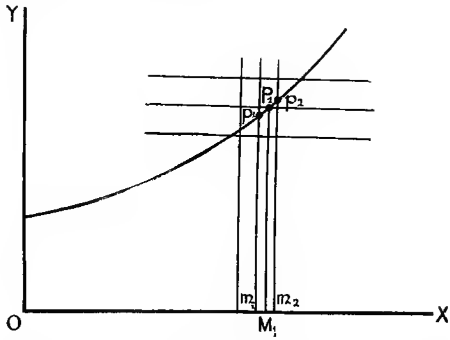

FIG. 47.

Now $e^{x+k}/e^x = e^k = 1 + k + \ldots > 1$, when k is any positive commensurable quantity. Hence y increases when x increases, and however closely we take the vertical lines of the network, the curve always rises to the right.

$$p_2 m_2 > P_1 M_1 > p_1 m_1;$$

$$\therefore \quad e^{x_1+\delta} > y_1 > e^{x_1}.$$

$$e^\delta > \frac{y_1}{e^{x_1}} > 1.$$

But $e^\delta = 1 + \delta + \dfrac{\delta^2}{2!} + \ldots < 1 + \delta + \delta^2 + \delta^3 + \ldots < \dfrac{1}{1-\delta}$;

$$\therefore \quad \frac{1}{1-\delta} > \frac{y_1}{e^{x_1}} > 1.$$

By increasing the fineness of the reticulation δ may be made as small, and therefore $1/1-\delta$ as near 1, as we please; and therefore y_1 may be made to differ from e^{x_1} by as little as we please.

Now write the equation in the form $x = \log_e y$.* Then if y_1 is any stated value of y, $\log_e y_1$ can always be identified as between x_1 and $x_1 + \delta$, where δ is as small as we please. The actual evaluation of logarithms is shown on pp. 127–8.

Extension of the Exponential Theorem.

If a and x are commensurable, $a^x = (e^{\log_e a})^x$, by definition of a logarithm.

Let $\log_e a$ be between the near commensurable quantities b and $b + \delta$, where δ is positive; then

$$e^{(b+d)x} > a^x > e^{bx}.$$

$$\therefore \quad a^x < 1 + x(b+\delta) + \frac{x^2}{2!}(b+\delta)^2 + \frac{x^3}{3!}(b+\delta)^2 + \ldots \text{(from p. 120)}$$

$$> 1 + x \, . \, b + \frac{x^2 b^2}{2!} + \frac{x^3 b^3}{3!} + \ldots$$

Thus, it is shown below that $\log_e 2 < \cdot 6932 > \cdot 6931$. Then 2^x is intermediate between the results obtained by writing $\log_e a = \cdot 6932$ and $\log_e a = \cdot 6931$.

This result is generally, but somewhat erroneously, written

$$\mathbf{a^x = 1 + x \log_e a + \frac{x^2}{2!}(\log_e a)^2 + \ldots + \frac{x^t}{t!}(\log_e a)t + \ldots .}$$

EXAMPLE. When $x = \frac{1}{10}$, write down the remainder after three terms in each of the two series, and hence approximate to $2^{\frac{1}{10}}$, showing that it is between 1·07171 and 1·07179.

An important limit.

$\underset{n \to \infty}{\mathrm{Lt}} \left(1 + \frac{x}{n}\right)^n = e^x$, when x is **positive** and finite, and n **integral.**

For $\left(1 + \frac{x}{n}\right)^n = 1 + n\frac{x}{n} + \frac{n(n-1)}{1 \, . \, 2}\frac{x^2}{n^2} + \ldots$ by the binomial theorem, n being taken greater than x,

$$= 1 + x + \frac{1 - \frac{1}{n}}{1 \, . \, 2} x^2 + \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{1 \, . \, 2 \, . \, 3} x^3 +$$

$$< 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \text{to } \overline{n+1} \text{ terms}, < e^x \text{ when } n \text{ is finite.}$$

* $\log y$ is often written for $\log_{10} y$ in elementary numerical work and for $\log_e y$ in theoretical analysis.

CONTENTS

SECTION V

SECTION VI

SECTION VII

SECTION VIII

SECTION IX

INDEX TO DEFINITIONS

CORRIGENDA

Page 7. 15th line from bottom. *For* M *read* N.

10. Ex. 2. *For* ·347 *read* ·00347 *and for* 2·5403 *read* ·5403.

55. Ex. 8. *For* 48 *read* 86.

79. 9th line. *For* cx^2+bx+a *read* ax^2+bx+c.

84. 3rd line from bottom. *For* positive *read* negative.

87. 15th line. *For* $= \alpha_n($ *&c. read* $= a_n($ *&c.*

98. 10th line. *For* S_{664} *read* S_{665}.

101. 11th line. *Delete* that.

19th line. *For* x between $x_1 \pm h$ *read* $0 < |x - x_1| \ngtr h$.

102. 14th line. *For* d *read* l.

104. 2nd line from bottom. *For* 12×10^{12} *read* $\frac{1}{3}$.

105. 2nd line. *Insert* L^t before last fraction.

107. 7th line. *Read* even or odd.

116. 5th line. In the second term *read* $[m_2]$ *for* $[m_1]$.

120. 2nd line from bottom. *For* x^2 *read* x_2.

6th line from bottom. *For* x^t *read* $x_1{}^t$.

121. 9th line. *For* 2 *read* n.

123. 17th line. *For* $(\log_e a)t$ *read* $(\log_e a)^t$.

135. 3rd line. *For* $\frac{A}{B}$ *read* $-\frac{A}{B}$.

140. 16th line. *For* (ξ, y) *read* (ξ, η).

143. Last paragraph. *For* $\frac{g^2}{a^2}+\frac{f^2}{b^2}$ *read* $\frac{g^2}{a}+\frac{f^2}{b}$ in the three lines where it occurs.

150. Ex. 9. *Read* without or within.

152. 11th line. *For* $(0, \pm \alpha e)$ *read* $(\pm \alpha e, 0)$.

153. 19th line. *For* $(0, p)$ *read* $(p, 0)$.

157. 7th line. *For* $\ngtr$ *read* $<$; and 9th line, *for* $\nless$ *read* $>$.

19th line. *Insert* , *after* ∞.

158. 4th line from bottom. *Read* The square of the length.

160. In the first five lines interchange the letters P and D.

189. 6th line. *For* $\sin^2\frac{\theta}{2}$ *read* $2\sin^2\frac{\theta}{2}$.

195. 6th line. *For* 70 *read* 69.

203. Last line. *For* zero *read* small compared with δx.

265. Last line. Insert $+1$ *after* $f(x_1 y_1 z_1)$.

In Section VI, pp. 156 *seq.* and 183 *seq.* *For the phrase* 'Intersection of the equations' *read* 'Intersection of the loci represented by the equations'.

1435 *Bowley, Pure Mathematics* *Face p.* xii
May, 1914

SECTION I

ALGEBRA

In the beginning of Algebra, arithmetical statements are generalized by the replacement of particular numerical cases by letters. In the simplest cases the letters stand for positive integers, but an extension of their meaning in two ways is suggested by the expressions obtained; on the one hand the ideas of negative quantities and of the product and ratio of two negative quantities are introduced, on the other a letter is connected with a physical or geometrical measurement.

The first of these extensions affords examples of the process, which is used frequently in the sequel, of interpreting the meaning of and then defining a quantity newly introduced, so that it is closely related to and obeys the same laws as quantities already known. E.g. it is shown as generalized arithmetic that if a, b, c, d are integers, and a, c greater respectively than b, d, then

$$(a-b)(c-d) = ac-bc-ad+bd.$$

This result is assumed to be true whatever quantities the letters stand for, and it is found that $-b \times -d$ must then equal $+bd$, and that no inconsistency is introduced if the definition thus suggested for the meaning of the product of two negative quantities is adopted. A definition arising in this way from a generalization of a law suggested by simple cases is known as a *mathematical convention*, that is, an interpretation of symbols agreed on by mathematicians.

The second extension, that letters should measure physical or geometrical quantities, leads to such expressions as $x = \sqrt{2}$ (the ratio of the diagonal of a square to its side) and $x = \pi$ (that of a circumference to a diameter), which cannot be expressed exactly as integers or as the ratio of two integers, though approximations such as $\frac{7}{5}$ and $3{\cdot}14 = 314/100$ can be found for them.* Quantities are said to be *commensurable** (see p. 25) when they can be expressed as n or n/m where n and m are integers, and *incommensurable* when they cannot be so expressed.

A final extension is that x may stand for a quantity not realizable in the physical universe, but subject to artificial rules, as is shown in Section VIII of this book; such quantities are termed *imaginary*, while those which can represent physical measurements are termed *real*.

The process of the following article illustrates the application of these extensions. To a law evidently true for certain letters standing for positive integers, the convention is applied that it shall also be true when the meaning of the letters is extended so that they stand for commensurable fractions and for negative quantities, and definitions are obtained. It is to be noticed that the particular convention used cannot readily be extended to include incommensurables.

Indices.

In elementary algebra $a^2 \times a^3 = (a \times a) \times (a \times a \times a) = a^{2+3}$.

Similarly, if m and n are any positive integers,

$$a^m \times a^n = a^{m+n}. \qquad \text{(i)}$$

Here m is called the *index* and a^m the m^{th} *power* of a.

The expression a^x has a defined meaning when x is a positive integer; it is reasonable to extend this definition so as to include cases where x has any value, as described on the previous page. The definitions universally adopted are obtained as in the following paragraphs from the *convention* that equation (i) shall be true when m and n are any commensurable quantities.

Definitions.

I. Let $x = \frac{p}{q}$, where p and q are positive integers.

$$\left(a^{\frac{p}{q}}\right)^q = a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \ldots (q \text{ factors}) = a^{\frac{p}{q}+\frac{p}{q}+\ldots q \text{ terms}} = a^{\frac{p}{q}\times q} = a^p.$$

$a^{\frac{p}{q}}$ is then defined as meaning any q^{th} root of a^p.

Arithmetical considerations show that there is one and *only*

* See Appendix, p. 269.

one real positive q^{th} root when a is any positive quantity. This is called the *principal* root.

[By drawing the graphs of $y = x^2$, $y = x^3$, $y = x^4$, &c., it can be seen that when the index is 2, 4, 6 ..., y is never negative, and for any assigned positive value of y there is one positive value of x and a numerically equal negative value; whereas if the index is 3, 5, 7 ..., y ranges through all negative values when x is negative and through all positive values when x is positive, so that for any assigned positive value of y there is one positive value of x. Hence if $x^q = y = a^p$, a and therefore y being positive, one real value of x exists, and this is the principal value as just defined.]

It is shown algebraically (p. 231) that q different roots, positive or negative, real or imaginary, can be identified and defined.

II. Let $x = 0$.

Then in accordance with equation (i), $a^0 \times a^n = a^{0+n} = a^n$, and $a^0 = 1$.

a^0 is then defined to be unity.

III. Let $x = -\frac{p}{q}$, where p and q are positive integers.

$$a^x \times a^{\frac{p}{q}} = a^{x+\frac{p}{q}} = a^0 = 1, \text{ and } a^x = \frac{1}{a^{\frac{p}{q}}} = \frac{1}{a^{-x}}.$$

In the case where $q = 1$, $a^x = \frac{1}{a^p}$. E.g. $a^{-4} = \frac{1}{a^4}$.

In other cases the meaning of $a^{\frac{p}{q}}$ is obtained from I.

E.g. $a^{-\frac{3}{4}}$ is unity divided by any fourth root of a^3.

a^x (x negative) is then defined to be the reciprocal of $a^{x'}$, where $x' = -x$.

A meaning has now been assigned to a^x for all positive and negative values of x which are integral or can be expressed as the ratio of two integers, i.e. for all commensurable values of x.

No meaning has yet been assigned to such expressions as $a^{\sqrt{3}}$ or $a^{\sqrt{-2}}$; for these see p. 249 below.

Division Rule.

If x_1 and x_2 are any commensurable quantities,

$$\mathbf{a}^{\mathbf{x}_1} \div \mathbf{a}^{\mathbf{x}_2} = a^{x_1} \times \frac{1}{a^{x_2}} = a^{x_1} \times a^{-x_2} = \mathbf{a}^{\mathbf{x}_1 - \mathbf{x}_2}, \quad . \quad . \quad \text{(ii)}$$

by equation (i), which has been extended to all commensurable indices.

Power Rule.

A. If x is any commensurable quantity and m a positive integer,

$(a^x)^m = a^x \times a^x \times \dots$ (m factors) $= a^{x+x+\dots(m \text{ terms})}$, by equation (i),
$= a^{x \times m}$.

B. If p and q are positive integers,

$$\left\{(a^x)^{\frac{p}{q}}\right\}^q = (a^x)^{\frac{p}{q} \times q} \text{ by A}, = (a^x)^p = a^{xp} \text{ by A}.$$

Similarly $\left(a^{\frac{xp}{q}}\right)^q = a^{\frac{xp}{q} \times q} = a^{xp}$.

$\therefore$ $(a^x)^{\frac{p}{q}}$ and $a^{x \cdot \frac{p}{q}}$ are q^{th} roots of the same quantity.

$\therefore$ the real positive, or principal, values of $(a^x)^{x_1}$ and $a^{x \times x_1}$ are equal when x is a positive commensurable fraction, and a is positive.

C. $(a^x)^0 = 1 = a^0 = a^{x \times 0}$.

D. If x_1 is a negative commensurable quantity,

$$(a^x)^{x_1} \times (a^x)^{-x_1} = (a^x)^{x_1 - x_1} = (a^x)^0 = 1\,;$$

$\therefore$ $(a^x)^{x_1} = \dfrac{1}{(a^x)^{-x_1}} = \dfrac{1}{a^{xx'}}$, where $x' = -x_1$ and is positive, if x' is integral (A), or if we deal only with principal roots (B),

$$= a^{-xx'} = a^{xx_1}.$$

A, *B*, *C*, *D* may be written in one statement thus:—If $\mathbf{x}$, $\mathbf{x}_1$ are any commensurable quantities,

$$(\mathbf{a}^{\mathbf{x}})^{\mathbf{x}_1} = \mathbf{a}^{\mathbf{x}\mathbf{x}_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

(α) when $\mathbf{x}_1$ is integral or zero, (β) when $\mathbf{x}_1$ is a commensurable fraction, $\mathbf{a}$ is positive, and the principal roots of both sides are taken.

Distributive Rule.

I. If n is a positive integer,

$$(ab)^n = (ab) \times (ab) \times \dots \text{ to } n \text{ factors}$$
$$= \{a \times a \times \dots \text{ to } n \text{ factors}\} \times \{b \times b \times \dots \text{ to } n \text{ factors}\}$$
$$= a^n b^n.$$

II. If p and q are positive integers,

$$\left\{(ab)^{\frac{p}{q}}\right\}^q = (ab)^{\frac{p}{q}\times q} \text{(by Power Rule)} = (ab)^p = a^p b^p \text{ by I,}$$

and $\left(a^{\frac{p}{q}} b^{\frac{p}{q}}\right)^q = \left(a^{\frac{p}{q}}\right)^q \left(b^{\frac{p}{q}}\right)^q$ by I $= a^p b^p$ by Power Rule;

$\therefore$ $(ab)^{\frac{p}{q}}$ and $a^{\frac{p}{q}} b^{\frac{p}{q}}$ are q^{th} roots of the same quantity, and are equal if a and b are positive and principal roots only are taken.

III. $(ab)^0 = 1$ and $a^0 b^0 = 1 \times 1 = 1$.

IV. If x_1 is a negative commensurable quantity $= -x'$,

$$(ab)^{x_1} = \frac{1}{(ab)^{x'}} = \frac{1}{a^{x'}.\,b^{x'}} \text{ by I or II (for principal roots),}$$
$$= a^{-x'} \times b^{-x'} = a^{x_1}.\,b^{x_1}.$$

Hence $$(\mathbf{ab})^{\mathbf{x}} = \mathbf{a}^{\mathbf{x}}.\,\mathbf{b}^{\mathbf{x}} \quad \dots \dots \quad \text{(iv)}$$

(i) when **x** is integral or zero, (ii) when **x** is a commensurable fraction, and **a** and **b** are positive, if principal roots are taken throughout.

It is now easily shown that under the same conditions

$$(abcd\dots)^x = a^x b^x c^x d^x \dots.$$

Note. The separation of terms in $(a+b)^n$ is performed by the Binomial Theorem (p. 23).

The relations (i), (ii), (iii), (iv) are the Rules of Indices.

Restricting all expressions involving fractional indices for the present to their principal roots, we can now represent the equation $y = a^x$ graphically, when a is positive.

To obtain the most useful graph we take $a = 10$.

By arithmetic processes $10^{\frac{1}{2}} = \sqrt{10} = 3{\cdot}162\dots$.

By rule (iii) $10^{\frac{1}{4}} = 10^{\frac{1}{2}\times\frac{1}{2}} = (\sqrt{10})^{\frac{1}{2}} = \sqrt{3{\cdot}162\dots} = 1{\cdot}778\dots$.

Similarly $10^{\frac{1}{8}} = \sqrt{1{\cdot}778} = 1{\cdot}334\dots$, $10^{\frac{1}{16}} = \sqrt{1{\cdot}334\dots} = 1{\cdot}155\dots$.

[Continuing this process we notice that $10^{\frac{1}{2^n}}$ approaches 1 as n is increased; this is of assistance in understanding that a^0 is properly defined as unity.]

$10^{\frac{3}{4}} = 10^{\frac{1}{2}} \times 10^{\frac{1}{4}}$ by rule (i), $= 3{\cdot}162\ldots \times 1{\cdot}778\ldots = 5{\cdot}623\ldots$,

$10^{1\frac{1}{4}} = 10 \times 10^{\frac{1}{4}} = 17{\cdot}78\ldots$, $10^{-\frac{1}{2}} = 10^{\frac{1}{2}-1} = \sqrt{10} \times \frac{1}{10} = {\cdot}3162$.

By such processes all the values in the following table can be obtained, and the eighths and sixteenths can also be calculated.

x	y	x	y	x	y	x	y	x	y
0	1	1	10	2	100	-1	·1	-2	·01
$\frac{1}{4}$	1·78...	$1\frac{1}{4}$	17·8...	$2\frac{1}{4}$	178...	$-\frac{3}{4}$	·178...	$-1\frac{3}{4}$	·0178...
$\frac{1}{2}$	3·16...	$1\frac{1}{2}$	31·6...	$2\frac{1}{2}$	316 ..	$-\frac{1}{2}$	·316...	$-1\frac{1}{2}$	·0316...
$\frac{3}{4}$	5·62...	$1\frac{3}{4}$	56·2...	$2\frac{3}{4}$	562...	$-\frac{1}{4}$	·562...	$-1\frac{1}{4}$	·0562...

Graph of Principal Values of $y = 10^x$, or $x = \log_{10} y$.

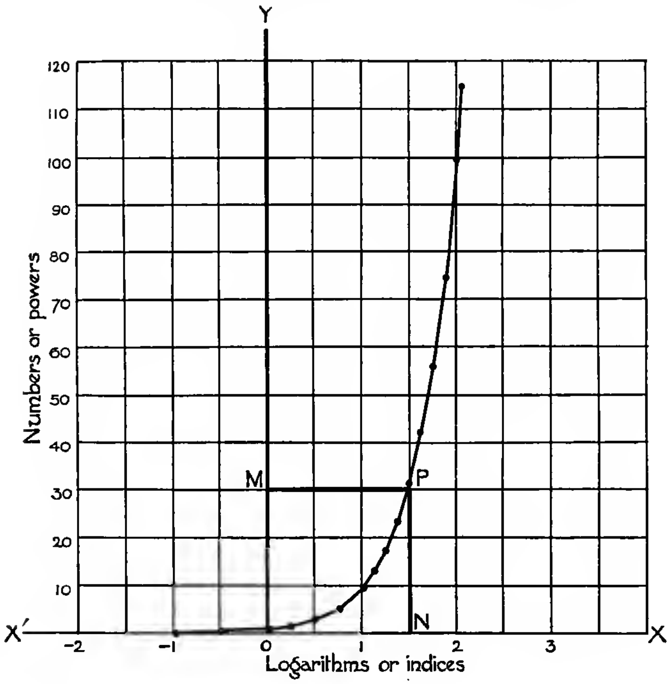

Fig. 1.

Examples.

[In these all powers are to be confined to their principal values.]

1. Show that $(12\,a^{-1})^{\frac{1}{2}} = \dfrac{2\sqrt{3}}{a^{\frac{1}{2}}}$.

2. Express in powers of prime factors $27^{\frac{2}{3}} \times 1728^{\frac{1}{4}}$.

3. Simplify $6^{\frac{1}{2}} + 24^{\frac{1}{2}}$.

4. Simplify $(4\,a^2\,b^{-3})^{\frac{1}{2}} \div (8\,a^3\,b^{-2})^{\frac{1}{3}}$.

5. Multiply $(a^{\frac{1}{2}} + b^{\frac{1}{2}})$ by $(a^{\frac{1}{2}} - b^{\frac{1}{2}})$.

6. Divide $a \pm b$ by $a^{\frac{1}{3}} \pm b^{\frac{1}{3}}$.

7. Given $10^{\frac{1}{3}} = 2{\cdot}154...$ and $10^{\frac{1}{2}} = 3{\cdot}162$, find $10^{\frac{1}{6}}$ and $10^{\frac{1}{12}}$, and fill in the table of 10^x when $x = \frac{1}{12}, \frac{2}{12}, \frac{5}{12}, \frac{7}{12}, \frac{8}{12}, \frac{10}{12}, \frac{11}{12}$, and thence when $x = 1\frac{5}{12}, -\frac{7}{12}$, &c. Then draw the graph of 10^x, using 37 values of x from $x = -1$ to $x = +2$.

$$(\text{e.g. } 10^{-\frac{5}{12}} = 10^{\frac{7}{12}-1} = 10^{\frac{1}{3}+\frac{1}{4}-1} = 10^{\frac{1}{3}} \times 10^{\frac{1}{4}} \div 10.)$$

Logarithms.

If $y = a^x$, then x is said to be the *logarithm* of y to base a, the principal root being taken when x is fractional. The equation is then written $x = \log_a y$. A positive quantity is always taken as base. At present we restrict ourselves for numerical and graphic illustration to the case $a = 10$.

If we assume that the curve representing 10^x (on p. 6) is continuous, i.e. that every line parallel to XX' (y being positive) intersects the curve, we can at once approximate to the logarithm of a number. Thus if $y = 30$ ($0M$ or NP), $x = 1{\cdot}48$ nearly ($0M$).

In general, when x is commensurable with unity and not an integer, y is incommensurable; so that when y is commensurable, no commensurable value of x can be found to satisfy $y = 10^x$, and we have as yet given no meaning to an incommensurable index. We cannot, therefore, find exact logarithms of commensurable numbers.

Practically, logarithms are used for approximate, not exact, calculations. Thus when the tables give $\log_{10} 2 = {\cdot}30103$, the meaning is that $10^{\cdot 30103}$ is nearer to 2 than is $10^{\cdot 30102}$ or $10^{\cdot 30104}$. Here all the indices are commensurable and their meaning has been assigned.

The determination of the logarithm of y is then the determination of a commensurable index x, such that 10^x differs from y by a quantity which is negligible in calculation.

Laws of Logarithms.

If $\log_a y_1 = x_1$ and $\log_a y_2 = x_2$, then $a^{x_1} = y_1$, $a^{x_2} = y_2$.

$\therefore\ y_1 \times y_2 = a^{x_1+x_2}$. (Rule (i), p. 2.)

$$\therefore\ \mathbf{\log_a y_1 y_2 = x_1 + x_2 = \log_a y_1 + \log_a y_2}. \quad . \quad . \quad \text{(i)}$$

Again $y_1 \div y_2 = a^{x_1-x_2}$. (Rule (ii), p. 4.)

$$\therefore\ \mathbf{\log_a \frac{y_1}{y_2} = x_1 - x_2 = \log_a y_1 - \log_a y_2}. \quad . \quad . \quad \text{(ii)}$$

Again $y_1{}^m = a^{mx_1}$, where m is commensurable, the principal values being taken. (Rule (iii), p. 4.)

$$\therefore\ \mathbf{\log_a y_1{}^m = mx_1 = m \log_a y_1}. \quad . \quad . \quad . \quad \text{(iii)}$$

If $\log_a b = u$ and $\log_b c = v$, then $a^u = b$ and $b^v = c$.

$$\therefore\ a^{uv} = b^v = c.$$

$$\therefore\ \mathbf{\log_a c = uv = \log_a b \times \log_b c}. \quad . \quad . \quad . \quad \text{(iv)}$$

In particular, if $c = a$, $\log_a b \times \log_b a = \log_a a = 1$.

Laws (i), (ii), and (iii) enable us, with the help of tables of logarithms, to replace the processes of multiplication, division, and root-extraction or raising to a power, by the easier and more rapid processes of addition, subtraction, and division or multiplication respectively. Law (iv) enables us to change from one base b to another base a, by multiplying the logarithms by a constant factor $\log_a b$.

Approximate Calculation of Logarithms.

Logarithmic tables have actually been calculated by the methods of more advanced algebra (see p. 127 below). The following approximations are given to illustrate the laws of logarithms, and to facilitate comprehension of the tables.

By actual multiplication it can be shown that 2^{63} is less than 10^{19}.*

$\therefore\ \log_{10} 2$ is less than $\frac{19}{63}$, less than $\cdot 3016$.

It can also be shown that 2^{196} is greater than 10^{59}.

$\therefore\ \log_{10} 2$ is greater than $\frac{59}{196}$, greater than $\cdot 30102$.

Hence $\log_{10} 2 = \cdot 301$ approx.

---

* $2^{20} = 1048576 < 105 \times 10^4$; $2^{60} = (2^{20})^3 < 116 \times 10^{16}$.

Again, 3^{13} is less than $2^4 \times 10^5$.

$\therefore$ 13 log 3 is less than 4 log 2 + 5 log 10. (Laws (i) and (iii).)

Hence log 3* is less than $\frac{1}{13}$ (1·2064 + 5), less than ·4775.

Again, 3^{21} is greater than 10^{10}, and log 3 is greater than ·4761.

Hence $\log_{10} 3 = \cdot 477$ approx.

By Law (i) log 6 = log 2 + log 3 = ·778 approx.

By Law (iii) log 4 = 2 log 2 = ·602, and log 8 = 3 log 2 = ·903 approx.

By Law (ii) log 5 = log 10 − log 2 = 1 − ·301 = ·699 approx.

By Law (iii) log 9 = 2 log 3 = ·954 approx.

We can now write down the logarithms of all integers whose prime factors are 2, 3, and 5. We can also approximate to other primes as we come to them.

The reader is left to show the correctness of the following table and to approximate to the numbers not filled in. The results should be compared with the graph on p. 6.

Numbers.	Logs.	Numbers.	Logs.	Numbers.	Logs.
1	0	11	—	21	—
2	.301	12	1.079	22	—
3	.477	13	—	23	—
4	.602	14	—	24	1.380
5	.699	15	1.176	25	1.398
6	.778	16	1.204	26	—
7	—	17	—	27	1.431
8	.903	18	1.255	28	—
9	.954	19	—	29	—
10	1.000	20	1.301	30	1.477

As an example of method, notice that $7^4 = 2401 = 2400$ nearly. Hence 4 log 7 = log 24 + 2 log 10 = 3·380, and log 7 = ·845 nearly. The logarithms of 14, 21, and 28 can then be written down.

Actual tables of logarithms, which are readily obtainable and are not given in this book, need oral explanation. They depend on the following principles, which are easily proved.

The integral part of a logarithm (called its *characteristic*) is determined by the position of the number in the scale $\ldots 10^4, 10^3, 10^2, 10^1, 1, 10^{-1}, 10^{-2} \ldots$. Thus if an integer consists of 4 digits, the characteristic of its logarithm is 3.

* When no base is given, the base 10 is implied in numerical calculations.

The decimal part of a logarithm (called its *mantissa*) depends only on the sequence of the digits in the number, e.g. log 3763, log 37·63, log ·3763 have the same mantissa, ·5755; their characteristics are 3, 1, and -1.

Only the mantissae are given in the tables.

The following examples show how simple calculations are carried out:

1. Multiply 37·63 by ·4752

$$\log 37{\cdot}63 = 1{\cdot}5755 \text{ from the tables.}$$

(Compare with the graph, p. 6.)

$$\log {\cdot}4752 = \bar{1}{\cdot}6769 \text{ from the tables.}$$

log (product) = sum of logs of factors = 1·2524 = log 17·88. Product is 17·88 (nearly).

2. Evaluate $\sqrt[5]{4{\cdot}785} \times \sqrt[3]{{\cdot}347} \div \sqrt{{\cdot}05823}$.

$$\tfrac{1}{5}\log 4{\cdot}785 = \tfrac{1}{5} \text{ of } {\cdot}6799 = {\cdot}1360$$

$$\tfrac{1}{3}\log {\cdot}347 = \tfrac{1}{3}(-3 + 2{\cdot}5403) = \bar{1}{\cdot}1801$$

$$\text{Sum} = \bar{1}{\cdot}3161$$

$$\tfrac{1}{2}\log {\cdot}05823 = \tfrac{1}{2}(-2 + {\cdot}7651) = \bar{1}{\cdot}3825$$

$$\text{Difference} = \bar{1}{\cdot}9336 = \log {\cdot}8582.$$

Result ·858.

3. $\log 83{\cdot}78^{12{\cdot}13} = 12{\cdot}13 \log 83{\cdot}78 = 1{\cdot}9231 \times 12{\cdot}13$

$$= 23{\cdot}33 = \log 2{\cdot}1 + 23.$$

Expression $= 2{\cdot}1 \times 10^{23}$ approx.

Only continual practice can make logarithmic computation safe. Great care is needed in considering to how many figures the answer can be stated accurately with the particular tables used.

Examples.

1. Given log 2 and log 3, write down log 120, log 125, and log 128. Hence find approximately log 121 and log 11, and hence log 22, log 33....

2. Find approximate values of log 13 from log 168, of log 17 from log 288 and log 300, and of log 19 from log 360.

3. From the results of the previous examples, complete the three-figure table of logarithms from log 1 to log 100, and compare with the graph and with the printed tables.

4. Find $\log_3 4$, $\log_7 10$, $\log_5 2$.

[$\log_3 4 = \log_{10} 4 \div \log_{10} 3$. Rule (iv).]

5. If P is the present value of C due in n years at r per cent. interest, $C = P \times \left(1 + \frac{r}{100}\right)^n$.

Use logarithms to find in what time a sum of money is doubled by compound interest at 4 per cent.

Find the present value of £478 10*s.* dne in 6 years at 5 per cent.

6. Given that 1 inch = 2·540 centimetres, find the number of acres in a hectare. [1 acre = 4840 sq. yards, 1 hectare = 10^4 sq. metres.]

7. Given that 1 lb. = 453·6 grammes, find the number of kilograms in a ton.

Inequalities.

The sign $\neq$ means not equal to,

$>$ „ greater than,

$\ngtr$ „ not greater than, i.e. equal to or less than,

$<$ „ less than,

$\nless$ „ not less than, i.e. equal to or greater than.

Algebraic quantities are either real, imaginary, or complex.

The following examples of inequalities apply only to real quantities, for which the fundamental statement is $x^2 \nless 0$ for all values of x.

The nature of imaginary and complex quantities is discussed below (Seetion VIII).

An inequality is a relation between real quantities involving any of the signs at the head of this section. Consideration shows at once that if $a < b$, then $a+c < b+c$, whether c is positive or negative; in particular $a-b < b-b$.

$$< 0.$$

Similarly with any of the other signs.

Thus a quantity can be taken across to the other side of an inequality by changing its sign, just as in the case of equations.

Hence if $a < b$, then $-b < -a$, and $-a > -b$.

Also if c is a positive quantity, $ca < cb$, since c only changes the scale of the inequality,

$$\therefore \quad -cb < -ca \text{ and } -ca > -cb.$$

Henee both sides of an inequality can be multiplied by a positive, but not by a negative, quantity without changing its sign.

Again, $$(a-b)^2 \nless 0;$$

$$\therefore\ a^2+b^2 \nless 2ab,$$

and $$(a+b)^2 \nless 4ab.$$

Let $a^2 = \alpha$ and $b^2 = \beta$;

$\therefore\ \alpha+\beta \nless 2\sqrt{\alpha\beta}$, where α and β are positive.

If α and β are not equal, this becomes $\frac{1}{2}(\alpha+\beta) > \sqrt{\alpha\beta}$ as on p. 16.

Similarly, $a^2+b^2+b^2+c^2+c^2+a^2 > 2ab+2bc+2ca$ unless $a=b=c$;

$$\therefore\ a^2+b^2+c^2 > ab+bc+ca,$$

$$a^2+b^2+c^2-ab-bc-ca > 0,$$

whether a, b, c are positive or negative.

If $a+b+c$ is positive, we have by multiplication,

$$a^3+b^3+c^3-3abc > 0, \text{ unless } a=b=c.$$

If $a^3=\alpha$, $b^3=\beta$, $c^3=\gamma$, this becomes

$$\tfrac{1}{3}(\alpha+\beta+\gamma) > \sqrt[3]{\alpha\beta\gamma}.$$

In all these cases, $>$ becomes $=$, if $a=b=c$.

Generally, we can show that

$$\frac{a_1+a_2+\ldots+a_n}{n} > \sqrt[n]{(a_1 a_2 \ldots a_n)}$$

if a_1, a_2 ... are positive and not all equal.*

For if any two of the quantities, say a_3, a_5, are unequal, let $b=\frac{1}{2}(a_3+a_5)$.

Then, since $(a_3+a_5)^2 > 4a_3a_5$, $b^2 > a_3a_5$, while $b+b=a_3+a_5$.

Hence, if a_3 and a_5 are replaced by b, b in both sides of the inequality, the magnitude of the left is unchanged and of the right is increased.

Hence, so long as any two of the quantities are unequal, we can increase the geometric mean without affecting the arithmetic mean by substitution.

Therefore, if n quantities have a given sum, say s, their product is not a maximum so long as any two are unequal.

* These expressions are known as the Arithmetic and Geometric Means of the n quantities (see p. 14).

But it is evident that there must be a maximum product for a given sum. Hence the maximum product is obtained, when

$$a_1 = a_2 = \dots = a_n = \frac{s}{n}, \text{ and equals } \left(\frac{s}{n}\right)^n;$$

$$\therefore \quad \left(\frac{s}{n}\right)^n > a_1 a_2 \dots a_n, \text{ when any of the } a\text{'s are unequal,}$$

and $\frac{s}{n} > \sqrt[n]{a_1 a_2 \dots a_n}$, the principal root being taken. Q.E.D.

If $a, b, c \dots$ be positive proper fractions,

$$1 > (1-a)(1-b)(1-c) \dots \quad > 1-(a+b+c+\dots).$$

For $\qquad (1-a)(1-b) = 1-a-b+ab \quad > 1-a-b;$

$$\therefore \quad (1-a)(1-b)(1-c) > (1-\overline{a+b})(1-c),$$

since by assumption $1-c$ is positive,

$$> (1-a-b-c) \text{ by the first case.}$$

Similarly, the inequality can be extended to any number of factors.

It is evident that the product is less than 1, since each factor is less than 1.

The symbol $|x|$ signifies the numerical value of x independent of its sign. Thus if x measured the height of the thermometer *above* freezing point, x might be positive or negative; but $|x|$ would measure the distance *from* freezing point whether above or below, and would be a number without sign.

If $x^2 > a^2$, it follows that $|x|$ is greater than $|a|$, but unless it is known that x is positive it does not follow that $x > a$.

Actually we have $\qquad x^2 - a^2 > 0,$

$$(x+a)(x-a) > 0;$$

$\therefore \quad (x+a)$ and $(x-a)$ are both positive or both negative.

If a is positive, the inequality is satisfied if $x > a$ or $x < -a$.

Similarly, if $(x-\alpha)(x-\beta) > 0$, where $\alpha > \beta$, it follows that $x > \alpha$, or $< \beta$, whether α and β are positive or negative.

Other inequalities will be proved as they are needed.

Examples.

1. Show that $(a^3-b^3)(a-b)$ is positive, unless $a = b$.
2. Find for what values of x, $x^2 + 3x$ is greater than 4.
3. Find the condition that $(a+b)(a^2+b^2) > a^3+b^3$ if a is positive and b negative.

4. If a and b are positive and $a > b$, show, graphically or otherwise, that $a^x > b^x$ and that $\log a > \log b$ when x is positive, and that $a^x < b^x$ if x is negative.

The Progressions.

The n terms $a,\ a+d,\ a+2d, \dots\ a+\overline{n-1}\,d$, where n is a positive integer and a and d are any quantities, are said to be in *Arithmetical Progression.*

Write l for the n^{th} term, so that $l = a+\overline{n-1}\,d$.

Let $S = a+(a+d)+\dots+(a+\overline{n-1}\,d)$, i.e. let S be the sum of n terms.

Then writing the series backwards,

$$S = l+(l-d)+(l-2d)+\dots+(l-\overline{n-1}\,d).$$

Adding these equations,

$$2S = (a+l)+(a+l)+\dots+(a+l) = n(a+l),$$

$$\mathbf{S = n \cdot \tfrac{1}{2}(a+l) = \tfrac{1}{2}n\,(2a+\overline{n-1}\,d)},$$

$\frac{1}{2}(a+l)$ is the average of the terms.

If a, m, c are in Arithmetic Progression, m is called the Arithmetic Mean between a and c,

$$c-m = d = m-a;$$

$$\therefore\ m = \tfrac{1}{2}(a+c).$$

The n terms $a,\ ar,\ ar^2, \dots\ ar^{n-1}$, where n is a positive integer, and a and r are any quantities, are said to be in *Geometric Progression.*

Let $S = a+ar+ar^2+\dots+ar^{n-1}$,

Then $Sr = ar+ar^2+ar^3+\dots+ar^{n-1}+ar^n$;

Subtracting, $S(1-r) = a-ar^n$;

$$\therefore\ \mathbf{S = a \cdot \frac{1-r^n}{1-r} = a \cdot \frac{r^n-1}{r-1}}.$$

r^n can readily be found by logarithms, when r and n are given.

If a, g, c are in Geometric Progression, $\dfrac{c}{g} = r = \dfrac{g}{a}$.

$g = \pm\sqrt{ac}$ is called the Geometric Mean between a and c; it is usually taken to be positive.

The logarithms of terms in G. P., viz. $\log a$, $\log a + \log r$, $\log a + 2\log r, \ldots$, are in A. P., the common difference being $\log r$.

The n terms $\frac{1}{k}, \frac{1}{k+d}, \frac{1}{k+2d}, \ldots \frac{1}{k+\overline{n-1}d}$ are said to be in *Harmonic Progression*, where n is a positive integer and k and d are any quantities.

No simple expression can be given for the sum of n terms.

If a, h, c are the first three terms,

$$\frac{1}{h} - \frac{1}{a} = k + d - k = k + 2d - (k+d) = \frac{1}{c} - \frac{1}{h},$$

$$c(a-h) = a(h-c),$$

$$\frac{a}{c} = \frac{a-h}{h-c} = \frac{h-a}{c-h}.$$

h, satisfying this equation, is defined as the *harmonic mean* between a and c.

Evidently, $$h = \frac{2ac}{a+c}.$$

Comparing the three means, we see that $h = \frac{2 \cdot g^2}{2m}$, i. e. $g^2 = hm$.

Hence g is the geometric mean between h and m.

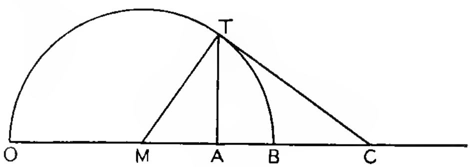

FIG. 2.

On diameter OB, centre M, describe a semicircle.

Take any point C on OB produced. Draw CT to touch the circle at T, and TA perpendicular to the diameter, meeting it at A. Join MT.

Then $CO - CM = CM - CB$,

$$CT^2 = CO \cdot CB.$$

Also $CT^2 = CA \cdot CM$, since CT is a tangent to the circle on diameter MT.

Hence CM, CT, and CA are the arithmetic, geometric, and harmonic means between CO and CB.

Also CM is greater than CT, which is greater than CA.

In this it is assumed that all the quantities involved are positive.

ADDENDUM. We have that the sum of the A.P.

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

The sums of $1^2 + 2^2 + \ldots + n^2$, and $1^3 + 2^3 + \ldots + n^3$ are also frequently needed.

The first can be obtained thus—

$$\begin{aligned} 3n^2 + 3n + 1 &= (n+1)^3 - (n)^3 \\ 3(n-1)^2 + 3(n-1) + 1 &= n^3 - (n-1)^3 \\ \cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ 3 \cdot 3^2 + 3 \cdot 3 + 1 &= 4^3 - 3^3 \\ 3 \cdot 2^2 + 3 \cdot 2 + 1 &= 3^3 - 2^3 \\ 3 \cdot 1^2 + 3 \cdot 1 + 1 &= 2^3 - 1^3. \end{aligned}$$

Adding, $3 \,.\, S_2 + 3 \,.\, S_1 + 1 \times n = (n+1)^3 - 1^3$, where S_1, S_2 are the sums of the first n integers and of their squares.

$$\therefore\ 3S_2 + 3 \,.\, \frac{n(n+1)}{2} + n = n^3 + 3n^2 + 3n,$$

$$\begin{aligned} S_2 = -\frac{n(n+1)}{2} + \frac{n(n+1)(n+2)}{3} &= \frac{n(n+1)}{6}\{-3 + 2(n+2)\} \\ &= \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

Similarly it can be shown from the identity $4n^3 + 6n^2 + 4n + 1 = (n+1)^4 - n^4$ that $4S_3 + 6S_2 + 4S_1 + n = (n+1)^4 - 1$, where S_3 is the sum of the cubes. After a little reduction this gives

$$S_3 = \left\{\frac{n(n+1)}{2}\right\}^2 = S_1^2.$$

EXAMPLES.

1. Sum the series $3 + 5 + 7 + \ldots + 23$.

2. ,, ,, $2 + 4 + 8 + \ldots + 1024$.

3. ,, ,, $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{3^{10}}$ correct to 6 decimal places, using logarithms.

4. Find the present value of an annuity of £100, the first payment to be made in 12 months from now, the second in 24 months, and the last in 20 years, reckoning compound interest at 4 per cent.

5. Sum the series $1^4 + 2^4 + \ldots + n^4$, and test the result when $n = 1, 2, 3, 4$.

6. Show that $1 \times 2 + 2 \times 3 + \ldots + n(n+1) = S_1 + S_2$ and write down its sum.

7. Sum $1 \times 2 \times 3 + 2 \times 3 \times 4 + \ldots + n(n+1)(n+2)$.

8. Sum $1 \times n + 2(n-1) + 3(n-2) + \ldots + n \times 1$.

Ratio and Proportion.

Ratio is that relation between two quantities, a and b, which is measured by the fraction $\frac{a}{b}$.

Four quantities, a, b, c, d, are in *proportion* if the ratio of a to b equals that of c to d; i. e. if $\frac{a}{b} = \frac{c}{d}$, and $\therefore\ ad = bc$.

Properties of Ratios.

Let $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n} = r$, so that $a_1 = rb_1$, $a_2 = rb_2$, &c.

Then $\frac{A_1 a_1 + A_2 a_2 + \ldots + A_n a_n}{A_1 b_1 + A_2 b_2 + \ldots + A_n b_n}$, where the A's are any quantities,

$$= \frac{A_1 rb_1 + A_2 rb_2 + \ldots +}{A_1 b_1 + A_2 b_2 + \ldots +} = r = \frac{a_1}{b_1} = \frac{a_2}{b_2} \quad . \quad . \quad . \quad \text{(i)}$$

Also $$\frac{A_1 a_1^2 + A_2 a_2^2 + \ldots}{A_1 b_1^2 + A_2 b_2^2 + \ldots} = \frac{r^2 (A_1 b_1^2 + A_2 b_2^2 + \ldots)}{A_1 b_1^2 + A_2 b_2^2 + \ldots} = r^2.$$

In particular, $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{\sqrt{a_1^2 + a_2^2}}{\sqrt{b_1^2 + b_2^2}}$, the positive roots being taken.

A very large number of similar relations, of more or less importance, can readily be proved.

If the ratios $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots$ are not equal, let r be the least of the ratios, and all the letters stand for positive quantities.

Let $\frac{a_1}{b_1} = r + d_1$, $\frac{a_2}{b_2} = r + d_2$, &c., where all the d's are positive except one, which is zero.

Then $\dfrac{a_1 + a_2 + \dots + a_n}{b_1 + b_2 + \dots + b_n} = \dfrac{b_1(r + d_1) + b_2(r + d_2) + \dots}{b_1 + b_2 + \dots}$

$$= r + \frac{b_1 d_1 + b_2 d_2 + \dots}{b_1 + b_2 + \dots} > r \text{ . (ii)}$$

By a similar method it can be shown that the fraction is less than the greatest of the ratios.

Corollary. $\dfrac{a+x}{b+x} > \dfrac{a}{b}$, if $a < b$, and $< \dfrac{a}{b}$ if $a > b$, a, b and x being positive.

Properties of Proportionals.

If $\dfrac{a}{b} = \dfrac{c}{d}$, then $\dfrac{a}{c} = \dfrac{b}{d} = \dfrac{a+b}{c+d} = \dfrac{a-b}{c-d}$, from the previous page, and

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}. \quad \dots \dots \dots \text{(iii)}$$

Also
$$\frac{a}{b} \pm 1 = \frac{c}{d} \pm 1;$$

$$\therefore \quad \frac{a \pm b}{b} = \frac{c \pm d}{d}.$$

If $\dfrac{a}{b} = \dfrac{b}{c}$, b is called the *mean proportional* between a and c; clearly $b^2 = ac$, and b is also the geometric mean.

Then $\left(\dfrac{a}{b}\right)^2 = \dfrac{a}{b} \times \dfrac{b}{c} = \dfrac{a}{c}$. From this property the ratio of a to c is said to be the duplicate ratio of a to b.

Let a straight line OB be bisected at M, and divided at A

Fig. 3.

and C, so that $MA \,.\, MC = MB^2$; then OB can be shown to be the harmonic mean between OA, OC.

For write a, b, c for MA, MB, MC.

Since $\quad ac = b^2, \ \dfrac{a}{b} = \dfrac{b}{c}$ and $\dfrac{a+b}{b-a} = \dfrac{b+c}{c-b}$;

$$\therefore \quad \frac{a+b}{b+c} = \frac{b-a}{c-b} = \frac{2b-(a+b)}{c+b-2b};$$

$\therefore$ $2b$, that is OB, is the harmonic mean between $(a+b)$, that is OA, and $(b+c)$, that is OC (p. 15).

Compare with this the figure on p. 15, whence it can be shown that CB, CA, CO are also in harmonic progression, if MA, MB, MC are in geometric progression.

Permutations and Combinations.

If there are n things, distinguishable one from another, the number of ways in which any number r of them can be chosen and arranged in order is called the number of the *permutations* of the n things r at a time, and is written ${}_nP_r$.

If we have r places, in order, to occupy with these things, we can occupy the first place with any one of them, that is in any of n ways. We have then $\overline{n-1}$ things left, with any of which we can fill the second place. Hence we can fill the first two places in any of $n \times (n-1)$ ways. Following out this plan, we have

$$\mathbf{{}_nP_r = n\,(n-1)\,(n-2)\ldots(n-r+1)}, \quad . \quad . \quad . \quad \text{(i)}$$

that is r factors.

This expression may be written $[n]_r$, the notation meaning the product of r successive factors, the first of which is n, and of the others each is 1 less than the preceding, whether n is integral or not.

If n factors are taken, n being a positive integer,

$$[n]_n = n\,(n-1)\ldots 3\,.\,2\,.\,1,$$

and this is written $\underline{|n}$, printed $n\,!$ and read as 'factorial n'.

We have then ${}_nP_r = [n]_r$ and ${}_nP_n = n\,!$

${}_nP_n$ is of course the number of ways in which n different things can be arranged in order, all being included.

Notice that $[n]_r = \dfrac{n!}{(n-r)!}$;

e.g. $(8)_3 = 8\,.\,7\,.\,6 = \dfrac{8\,.\,7\,.\,6\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1}{5\,.\,4\,.\,3\,.\,2\,.\,1} = \dfrac{8!}{5!} = \dfrac{8!}{(8-3)!}$.

If we are only concerned with the things which are chosen, and not with their order, the number of different groups, each containing r things, which can be chosen out of n distinguishable things is called the number of *combinations* of the n things r at a time and is written ${}_nC_r$.

When all possible different groups of r things have been chosen in succession and each has been arranged within itself in all possible ways, we have made all the possible permutations of the n things r at a time, and no permutation has been taken more than once.

But the number of groups is ${}_nC_r$, each can be arranged in ${}_rP_r$ ways, and therefore ${}_nC_r \times {}_rP_r$ different arrangements can be made.

$$\therefore \quad {}_nC_r \times {}_rP_r = {}_nP_r.$$

$$\therefore \quad {}_nC_r \times r! = [n]_r.$$

$$\therefore \quad {}_n\mathrm{C}_r = \frac{[\mathrm{n}]_r}{\mathrm{r}!} = \frac{\mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)\ldots(\mathrm{n}-\mathrm{r}+1)}{\mathrm{r}(\mathrm{r}-1)\ldots 2.1} = \frac{\mathrm{n}!}{(\mathrm{n}-\mathrm{r})!\ \mathrm{r}!} \quad \text{(ii)}$$

The argument can be readily followed by considering four things, a, b, c, d, taken three at a time. Here ${}_3P_3 = 6$, ${}_4P_3 = 24$, ${}_4C_3 = 4$.

6 Permutations.

4 Combinations	*abc, acb, bac, bca, cab, cba* *bcd, bdc, cbd, cdb, dbc, dcb* *cda, cad, dca, dac, acd, adc* *dab, dba, adb, abd, bda, bad*	6×4 permutations in all.

If r things are chosen $(n-r)$ things are left. Hence the number of groups n things r at a time that can be chosen equals the number of things $\overline{n-r}$ at a time that can be left.

$$\therefore \quad {}_nC_r = {}_nC_{n-r}.$$

Define ${}_nC_0$ as 1, corresponding to the one way in which all can be left, and take $0!$ to be 1, since we should have

$$1 = {}_nC_0 = \frac{n!}{(n-0)!\ 0!} = \frac{n!}{n!\ 0!} = \frac{1}{0!}$$

if the formula is to be unchanged when $r = 0$.

We have then such tables as

$${}_5C_5 = {}_5C_0 = 1, \qquad {}_6C_6 = {}_6C_0 = 1,$$

$${}_5C_4 = {}_5C_1 = \frac{5}{1} = 5, \qquad {}_6C_5 = {}_6C_1 = \frac{6}{1} = 6,$$

$${}_5C_3 = {}_5C_2 = \frac{5.4}{1.2} = 10, \qquad {}_6C_4 = {}_6C_2 = \frac{6.5}{1.2} = 15,$$

$${}_6C_3 = \frac{6.5.4}{1.2.3} = 20.$$

These can easily be generalized for odd and for even values of n.

The formula $${}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r$$
is so important that two proofs are given.

(1) $${}_nC_r + {}_nC_{r-1} = \frac{[n]_r}{r\,!} + \frac{[n]_{r-1}}{(r-1)\,!} = \frac{[n]_{r-1}(\overline{n-r+1}+r)}{r\,!}$$
$$= \frac{(n+1)\,[n]_{r-1}}{r\,!} = \frac{[n+1]_r}{r\,!} = {}_{n+1}C_r.$$

(2) Let there be $\overline{n+1}$ things. Place one, A, by itself, and the rest, n, in a group.

Consider in how many ways r can be chosen.

All the r can be taken from the group; that is, ${}_nC_r$ choices can be made, none of which include A.

Or A can be included and the remaining $\overline{r-1}$ chosen from the group in ${}_nC_{r-1}$ ways.

But the choice must either exclude or include A.

$$\therefore \quad {}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r.$$

As a special case, ${}_{n+1}C_1 = {}_nC_1 + {}_nC_0$.

By means of this formula we can construct the following table, which can readily be extended.

Values of ${}_nC_r$.

n	$r=0$	1	2	3	4	5	6	7	8	9	10	Sum of lines.
1	1	1										$2 = 2^1$
2	1	2	1									$4 = 2^2$
3	1	3	3	1								$8 = 2^3$
4	1	4	6	4	1							$16 = 2^4$
5	1	5	10	10	5	1						$32 = 2^5$
6	1	6	15 +	20	15	6	1					$64 = 2^6$
				‖								
7	1	7	21	35	35	21	7	1				$128 = 2^7$
8	1	8	28	56	70	56	28	8	1			$256 = 2^8$
9	1	9	36	84	126	126	84	36	9	1		$512 = 2^9$
10	1	10	45	120	210	252	210	120	45	10	1	$1024 = 2^{10}$

Here, for example, ${}_7C_3 = {}_6C_3 + {}_6C_2 = 20 + 15 = 35$.

Notice that $${}_nC_2 = \frac{n\,(n-1)}{2} = \frac{n-1}{2}(1 + \overline{n-1})$$
$$= \text{sum of the progression } 1 + 2 + 3 + \ldots\, n.$$

Binomial Theorem.—Positive Integral Index.

From the last paragraph we can at once proceed to the first case of the Binomial Theorem, which expresses $(a+x)^n$ as the sum of terms of ascending powers of x.

The case now taken is when n is a positive integer, a and x being any algebraic quantities.

We obtain the expression required by multiplying $a+x$ by $a+x$ again and again, and showing that the resulting coefficients are formed by the same rule as that which gives the values of ${}_nC_r$ as in the table on p. 21.

Thus $(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$. . . . . (α)

Multiply by $(a+x)$

$$\begin{array}{l} a + \quad x \\ \hline a^4 + 3a^3x + 3a^2x^2 + ax^3 \\ \qquad a^3x + 3a^2x^2 + 3ax^3 + x^4 \\ \hline \end{array}$$

$$(a+x)^4 = a^4 + (3+1)a^3x + (3+3)a^2x^2 + (1+3)ax^3 + x^4$$

$$= a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4 \quad . \; . \; . \; . \; . \quad (\beta)$$

Multiply by $(a+x)$

$$\begin{array}{l} a + \quad x \\ \hline \end{array}$$

$$(a+x)^5 = a^5 + (4+1)a^4x + (6+4)a^3x^2 + (4+6)a^2x^3 + (1+4)ax^4 + x^5$$

$$= a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5 \quad . \quad (\gamma)$$

Now compare the lines (α), (β), (γ) with the lines for $n = 3$, 4, 5 in the table for ${}_nC_r$, and it is seen that the coefficients of the various terms are formed by the same rule as is used for obtaining the numbers in the table; viz. that each coefficient is given by the sum of the coefficient written above it in the previous line and of the coefficient next before it. Also the coefficients in (α), 1, 3, 3, 1 are the numbers when $n = 3$.

Hence the coefficients in the expansions of successive powers of $(a+x)$ are the numbers given in the successive lines in the table.

Hence the coefficient of the $\overline{r+1}^{\text{th}}$ term in the expansion of

$(a+x)^n$ is the $\overline{r+1}^{\text{th}}$ number in the n^{th} line, that is ${}_nC_r$, that is $\frac{[n]_r}{r!}$.

The process of multiplication shows that there are $\overline{n+1}$ terms in the expansion of $(a+x)^n$, and that the indices of a fall from n to 0, while those of x rise from 0 to n.

Hence

$$(a+x)^n = a^n + n \cdot a^{n-1} x + \frac{n(n-1)}{1 \cdot 2} a^{n-2} x^2 + \ldots$$

$$+ \frac{n(n-1)(n-2)\ldots(n-r+1)}{1 \cdot 2 \cdot 3 \ldots r} a^{n-r} x^r + \ldots + n \cdot ax^{n-1} + x^n$$

$$= \sum_{r=0}^{r=n} \frac{n!}{(n-r)!\, r!} a^{n-r} x^r, \quad . \quad . \quad . \quad (1)$$

n being a positive integer.

The last written symbol means that for r we are to write in succession $r = 0, 1, 2, \ldots n$, and that the resulting terms are to be summed.

$\frac{n!}{(n-r)!\, r!} a^{n-r} x^r$ is called the *general term* of the expression.

As an example notice that if $a = x = 1$, we have

$$2^n = 1 + {}_nC_1 + {}_nC_2 + \ldots {}_nC_n;$$

hence the results in the last column of the table on p. 21.

This may also be obtained as follows: If there are n things at choice, we may take 0 in ${}_nC_0$ ways, or 1 in ${}_nC_1$ ways, or 2 in ${}_nC_2$ ways, &c., i.e. deal with them in ${}_nC_0 + {}_nC_1 + \ldots + {}_nC_n$ ways. We may also take or leave the first (2 ways), then take or leave the second (2×2 ways of dealing with the first two), and so on. Hence 2^n and ${}_nC_0 + {}_nC_1 + \ldots + {}_nC_n$ must be different ways of expressing the same number.

Before dealing with the Binomial Theorem when n is not a positive integer, we have to take the theory of limits.

Apart from their use for the Binomial Theorem here and on p. 115, Permutations and Combinations are mainly important in the theories of Chance and Probability, subjects of great importance in Statistics, but beyond the scope of this book.

Examples.

1. How many numbers can be formed from the digits 4, 5, 6, 7, using all or any, but none more than once in one number?

2. How many different signals can be made by altering the order of 8 distinctive flags? How many flags are necessary for expressing all the letters of the alphabet in this manner?

3. Find approximately $(1{\cdot}01)^6$ and $(103)^8$

[E. g. $11^4 = 10^4(1 + \frac{1}{10})^4$
$= 10^4(1 + 4 \times 10^{-1} + 6 \times 10^{-2} + 4 \times 10^{-3} + 1 \times 10^{-4})$
$= 1{\cdot}4641 \times 10^4$].

4. Expand, that is, express as a sum of terms, $\left(x - \dfrac{1}{x}\right)^6$.

5. Write down the general term and the 10th term of $\left(x^2 + \dfrac{2}{x}\right)^{12}$.

SECTION II

GEOMETRY

Similar Plane Figures.

It is assumed that the length of a straight line is capable of measurement, and can be expressed by a symbol, which can be manipulated by algebraic rules.

We shall deal with the ratio of lines, and the difficulty will at once arise that we cannot assume that the ratio can be expressed in the form $\frac{p}{q}$ where p and q are positive integers, unless the lengths are constructed to be respectively p and q times some unit length. The ratio of the diagonal of a square to its side is $\sqrt{2}:1$, and no exact common submultiple can be found of these lines.

Quantities which cannot be expressed exactly as multiples of the same unit are said to be incommensurable with each other. A number, such as $\sqrt{2}$, which cannot be exactly expressed as a multiple of or fraction of unity, is said to be incommensurable with unity.

Incommensurables can always be placed as intermediate between two commensurables. Thus $(1{\cdot}41422)^2 > 2 > (1{\cdot}41421)^2$; $\sqrt{2}$ is thus between 1·41421 and 1·41422 and is said then to be *evaluated*. Mathematical tables in general contain evaluations of incommensurables. The diagonal of a metre square, measured 'correctly to a millimetre' is 1·414 m. The presence of incommensurability causes no difficulty in practical measurement.

In dealing with such quantities, and later in the theory of limits (pp. 98 seq.), we shall need the following *axiom*:

Numerical quantities are equal to one another, if it can be shown that they differ by less than the smallest quantity we can assign.

E.g. the difference between 1 and $\cdot\dot{9}$ is less than ·000 ... 01, i.e. less than $\frac{1}{10^n}$, where n is as great an integer as we can name. If we assign 10^{-20} (or any other such quantity), 1 and $\cdot\dot{9}$ differ by less than it. The axiom then states that $1 = \cdot\dot{9}$.

PROPOSITION I. *The ratio of areas of triangles of equal altitude equals the ratio of their bases.*

Let ABC, DEF have equal altitudes, and let the base DE be greater than the base AB.

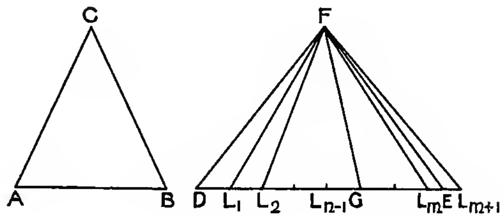

FIG. 4.

From DE cut off DG equal to AB, and join FG.

Then the areas DGF, ABC are equal.

Suppose DG divided into any number, n, of equal parts, EL_1, $L_1L_2, \dots L_{n-1}G$. From GE produced cut off successive intervals GL_{n+1}, $L_{n+1}L_{n+2}, \dots$, each equal to DL_1. Let E fall either at the point L_m or between the points L_m and L_{m+1}.

Join $L_1, L_2, \dots L_{m+1}$ to F. Then all the triangles FDL_1, $FL_1L_2 \dots$, $FL_{n-1}G, \dots FL_mL_{m+1}$ are equal in area. Call the area of any one of them a.

The area of DGF is $a \times n$, and that of DEF is equal to $a \times m$, or between $a \times m$ and $a \times (m+1)$.

$$\therefore \frac{\text{Area } DEF}{\text{Area } ABC} = \frac{\text{Area } DEF}{\text{Area } DGF} = \frac{ma}{na} = \frac{m}{n},$$

or is between $\frac{m}{n}$ and $\frac{m+1}{n}$;

and $\frac{DE}{AB} = \frac{DE}{DG} = \frac{DL_1 \times m}{DL_1 \times n} = \frac{m}{n}$, or is between $\frac{m}{n}$ and $\frac{m+1}{n}$.

But by taking n sufficiently large we can make the difference between $\frac{m}{n}$ and $\frac{m+1}{n}$, i.e. $\frac{1}{n}$, less than any quantity that can be assigned, without affecting the argument.

Then the ratio of the areas of the triangles and the ratio of their bases, DE and AB, differ from $\frac{m}{n}$, and therefore from each other, by less than any quantity that can be assigned.

$\therefore$ by the axiom on p. 25 the ratio of the areas equals the ratio of the bases. Q. E. D.

Proposition II. *If a number of parallel lines, AA', BB', CC', meet one straight line at A, B, C, ..., and another at A', B', C', ..., then*

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \ldots.$$

If the lines AD and $A'D$ are parallel, each ratio equals 1.

If the lines are not parallel, let them meet at O.

Join $A'B$, AB'.

Then $\frac{AO}{AB} = \frac{\text{Area } AOA'}{\text{Area } ABA'}$

by Prop. I $= \frac{\text{Area } AOA'}{\text{Area } AB'A'}$,

since the triangles ABA', $AB'A'$ are between the same parallels,

$$= \frac{A'O}{A'B'} \text{ by Prop. I.}$$

$$\therefore \quad \frac{AB}{A'B'} = \frac{AO}{A'O}$$

$$= \frac{OA+AB}{OA'+A'B'} \text{ (p. 18)} = \frac{OB}{OB'}.$$

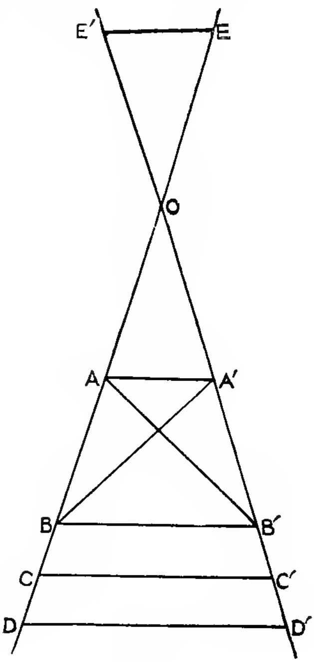

Fig. 5.

Similarly, it can be shown that $\frac{BC}{B'C'} = \frac{OB}{OB'}$.

$$\therefore \quad \frac{AB}{A'B'} = \frac{BC}{B'C'}. \qquad \text{Q. E. D.}$$

Similarly, $\dfrac{BC}{B'C'} = \dfrac{CD}{C'D'}$, &c.

The case where O is between two of the points concerned (as D and E in the figure) is left to the student.

Conversely, if $\dfrac{AB}{A'B'} = \dfrac{AO}{A'O}$, the lines AA' and BB' are parallel.

It does *not* follow that, if $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$, without further data, the lines AA', BB', CC' are parallel.

DEFINITION. *Two triangles, ABC, $A'B'C'$, are said to be similar if $\angle A = \angle A'$, $\angle B = \angle B'$, and consequently $\angle C = \angle C'$.*

PROPOSITION III. *If ABC, $A'B'C'$ are similar triangles, so that $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$, then shall*

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}.$$

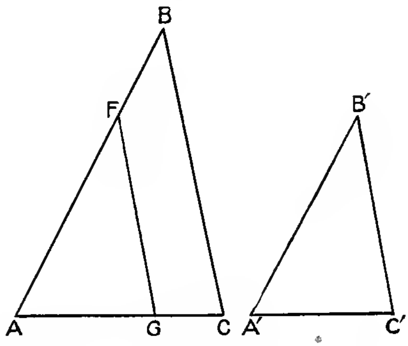

FIG. 6.

If $AC = A'C'$, the triangles are equal in all respects and each ratio is unity.

Let $AC > A'C'$. Apply $A'B'C'$ to ABC so that A' lies on A and $A'C'$ along AC, and let G be the point where C' lies.

Then $A'B'$ will lie along AB, since $\angle A' = \angle A$. Let B lie at F.

Then AFG equals $A'B'C'$ in all respects.

$\angle AFG = \angle B' = \angle B$ by hypothesis.

$\therefore$ FG is parallel to BC.

$$\therefore \frac{AC}{A'C'} = \frac{AC}{AG} = \frac{AB}{AF} \text{ (Prop. II)} = \frac{AB}{A'B'}.$$

Similarly, $\dfrac{AC}{A'C'} = \dfrac{BC}{B'C'}$ (by applying the point C' to C).

Q. E. D.

PROPOSITION IV. *If in two triangles ABC, $A'B'C'$,*

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} \text{ and } \angle A = \angle A',$$

the triangles are similar.

For apply $B'A'C'$ to BAC as in Prop. III,

$$\frac{AB}{AC} = \frac{A'B'}{A'C'} \text{ (by hypothesis)} = \frac{AF}{AG}.$$

$\therefore$ FG and BC are parallel.

$\therefore$ $\angle ABC = \angle AFG$ (Prop. II) $= \angle A'B'C'$ (by construction).

Similarly, $\angle ACB = \angle A'C'B'$.

$\therefore$ the triangles are similar (Definition). Q. E. D.

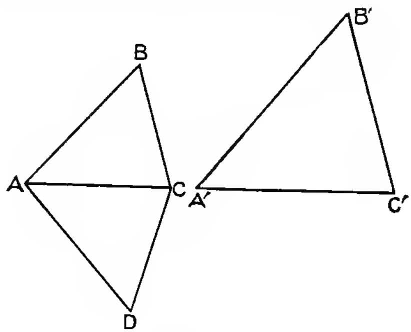

FIG. 7.

PROPOSITION V. *If in two triangles, ABC, $A'B'C'$,*

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'},$$

the triangles are similar. (Converse to Prop. III).

At A, C in AC on the opposite side to B make the angles CAD, ACD equal respectively to $\angle A'$ and $\angle C'$.

Then ACD and $A'C'B'$ are similar.

$$\frac{AD}{A'B'} = \frac{AC}{A'C'} \text{ (Prop. III) } = \frac{AB}{A'B'} \text{ (by hypothesis).}$$

$$\therefore \quad AD = AB.$$

Similarly, $\quad CD = CB.$

$\therefore$ the triangles ABC, ACD have equal angles (Eucl. i. 8).

$\therefore$ $\angle BAC = \angle CAD = \angle B'A'C'$ (by construction),

and $\angle BCA = \angle ACD = \angle B'C'A'$.

$\therefore$ ABC and $A'B'C'$ are similar triangles.

NOTE. *Relation between equality of triangles and similarity of triangles.*

Name the angles of a triangle A, B, C, and the sides opposite to them a, b, c.

(i) All triangles constructed with given C, a, and b are equal in all respects.

All triangles constructed with given C and given ratio $a:b$ are similar (Prop. IV).

(ii) All triangles constructed with given A, B (and $\therefore$ C) and a are equal in all respects.

All triangles constructed with given A, B (and $\therefore$ C) are similar (Definition).

(iii) All triangles constructed with given a, b, c are equal in all respects.

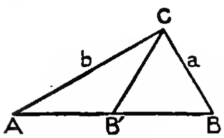

FIG. 8.

All triangles constructed with given ratios $a:b:c$ are similar (Prop. V).

(iv) If a, b, A are given, the triangles which can be constructed are equal to one or other of two triangles as ABC, or ACB'.

The corresponding case is, if A and the ratio $a:b$ are given, triangles are similar to one or other of two triangles.

The proof is left to the student.

The numerators AB, BC, CA in (e.g.) Prop. V are said to be *homologous* or *corresponding* with the denominators $A'B'$, $B'C'$, $C'A'$ respectively.

If r is the common ratio, each line equals its *homologue* multiplied by r; r may be called the linear magnification. This does not imply that r is always > 1.

PROPOSITION VI. *The ratio of the areas of two similar triangles ABC, $A'B'C'$ is equal to the square of the ratio of any pair of their corresponding sides, that is to r^2, where r is the linear magnification.*

Draw BM, $B'M'$ perpendicular to AC, $A'C'$.

The area of a triangle is numerically equal to half the product of the numbers measuring its base and altitude.

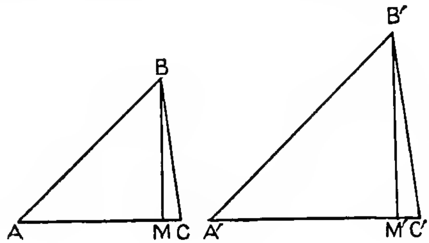

FIG. 9.

Since $\angle A = \angle A'$ (by hypothesis), and $\angle AMB = \angle A'M'B'$ (by construction), ABM and $A'B'M'$ are similar (Definition).

$$\therefore \quad \frac{BM}{B'M'} = \frac{AB}{A'B'} = r$$

$$\frac{\text{Area } ABC}{\text{Area } A'B'C'} = \frac{\frac{1}{2}AC \, . \, BM}{\frac{1}{2}A'C' \, . \, B'M'} = \frac{AC}{A'C'} \cdot \frac{BM}{B'M'} \, .$$

$$= r \times r = r^2 = \left(\frac{AC}{A'C'}\right)^2 = \left(\frac{AB}{A'B'}\right)^2 = \left(\frac{BC}{B'C'}\right)^2 .$$

DEFINITION. *Two plane polygons, each of n sides, $ABCD \ldots$ $A'B'C'D' \ldots$ are said to be similar if $\angle A = \angle A'$, $\angle B = \angle B' \ldots$, and $AB : A'B' = BC : B'C' = CD : C'D' \ldots$.*

The following construction shows that if $\overline{n-2}$ angles in the one equal respectively $\overline{n-2}$ angles in the other, and if the ratios of $\overline{n-1}$ pairs of corresponding sides are equal, then the remaining angles in the one equal respectively the remaining angles in the other, and the ratio of the remaining pairs of sides equals the ratio of any other corresponding pair.

Let $ABCDE$ be any polygon (taken as five-sided for illustration).

Draw a line $A'B'$, and let $\frac{A'B'}{AB} = r$.

At B' make an angle $A'B'C'$ equal to $\angle B$, and cut off $B'C' = rBC$.

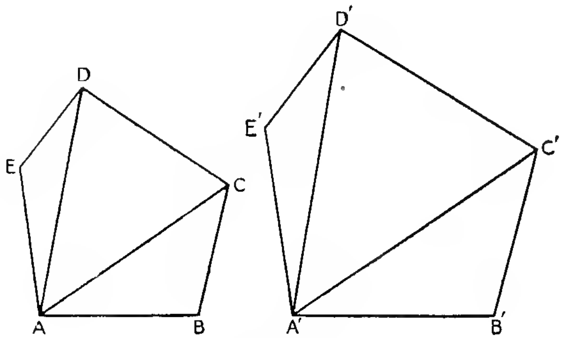

FIG. 10.

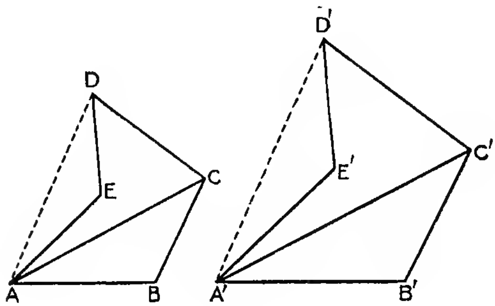

FIG. 11.

At C' make $\angle B'C'D' = \angle C$, and $C'D' = r \,.\, CD$.

At D' make $\angle C'D'E' = \angle D$, and $D'E' = r \,.\, DE$.

Join $A'E'$.

In our construction we have taken 4, $(5-1, n-1)$, ratios equal in the two figures and 3, $(5-2, n-2)$, angles equal.

We have now to show that $\angle E' = \angle E$, $\angle A' = \angle A$, and $E'A' = r \,.\, EA$.

Join AC, $A'C'$. By Prop. IV, ABC and $A'B'C'$ are similar, and $\therefore \angle ACB = \angle A'C'B'$ and $A'C' = r \,.\, AC$.

But $\angle BCD = \angle B'C'D'$; $\therefore$ by subtraction $\angle ACD = \angle A'C'D'$.

Also $\dfrac{D'C'}{DC} = \dfrac{C'A'}{CA} = r$. $\therefore$ the triangles ACD and $A'C'D'$ are similar (Prop. IV).

$\therefore A'D' = r \,.\, AD$. Also $\angle ADC = \angle A'D'C'$, and their differences from the equal angles CDE, $C'D'E'$, namely $\angle ADE$ and $\angle A'D'E'$, are equal.

But $D'E' = r \,.\, DE$. $\therefore$ the triangles ADE, and $A'D'E'$ are similar (Prop. IV).

$\therefore EA = r \,.\, E'A'$, $\angle A'E'D' = \angle AED$, and $\angle D'A'E' = \angle DAE$.

Looking again at the triangles ABC, ACD, AED and those similar to them, we have

$$\begin{aligned} \angle E'A'B' &= \angle B'A'C' + \angle C'A'D' \pm \angle D'A'E' \\ &= \angle BAC + \angle CAD \pm \angle DAE = \angle EAB. \end{aligned}$$

It is left as an exercise to show that the proof applies to polygons with any number of sides, and (with proper choice of signs) to any shape, re-entrant or not.

It should now be obvious that any line joining two angular points in the one polygon equals r times the line joining the corresponding points in the other.

PROPOSITION VII. *The ratio of the areas of two similar polygons is equal to the square of the ratio of corresponding sides.*

For the area $ABCDE$ = area $(ABC + ACD \pm ADE)$

$$= r^2 \times \text{area } (A'B'C' + A'C'D' \pm A'D'E') \quad \text{(Prop. VI)}$$

$$= r^2 \times \text{area } A'B'C'D'E'.$$

Similarly with polygons of any number of sides.

Similar plane figures are now seen to be like in shape, but differing in size. Any line in one equals the corresponding line in the other multiplied by r, and any area in one equals the corresponding area in the other multiplied by r^2. r^2 may be called the areal magnification.

These statements apply equally to similar curvilinear figures, if these are regarded as limiting forms of polygons consisting of a great number of very small sides (see pp. 65–6). Thus circles are all similar, the ratio of the circumference to diameter is constant, and the areas of two circles are in the ratio of the squares on their diameters.

These ideas can be extended to similar solids in which, if corresponding lines are in the ratio r, corresponding areas are in the ratio r^2, and corresponding volumes in the ratio r^3; but the analysis is beyond the scope of this book.

EXAMPLES.

1. If the vertical angle B of a triangle ABC (where $AB \neq BC$) is bisected internally and externally by lines meeting AC in D and E, then

$$\frac{AD}{DC} = \frac{AB}{BC} = \frac{AE}{CE}.$$

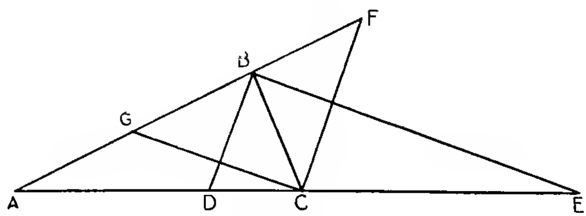

FIG. 12.

For draw CF parallel to DB to meet AB produced at F.

$\angle BCF = \angle CBD$ (since CF and BD are parallel) $= \angle ABD$ (const.)
$= \angle AFC$ " " "

$\therefore BF = BC$ (Eucl. i. 5).

Since DB and CF are parallel, $\dfrac{AD}{DC} = \dfrac{AB}{BF}$ (Prop. II) $= \dfrac{AB}{BC}$.

Now draw CG parallel to BE.

$\angle BCG$ = alternate angle $CBE = \angle EBF$ (const.) = internal angle BGC.

$\therefore BG = BC$ (Eucl. i. 5).

Since CG and BE are parallel, $\frac{AE}{EC} = \frac{AB}{BG}$ (Prop. II) $= \frac{AB}{BC}$.

[Notice that AD, AC, AE are in harmonic progression. See p.15.]

Corollary. If a point P moves so that the ratio, $\frac{AP}{PC}$, of its distances from two fixed points A and C is constant, its locus is a circle whose diameter is DE, where D and E are points in AC and AC produced such that $\frac{AD}{DC} = \frac{AE}{EC} = \frac{AP}{PC}$.

For the converse theorem, that if $\frac{AD}{DC} = \frac{AB}{BC}$, then BD bisects $\angle ABC$ and, if $\frac{AE}{EC} = \frac{AB}{BC}$, BE bisects $\angle ABC$ externally, is easy to prove.

$\therefore$ P, which now replaces B, is such that PD, PE bisect adjacent angles; $\angle DPE$ is a right angle and P lies on a circle whose diameter is DE.

---

2. If OT is a tangent to, and OAB, OCD secants of a circle $ATBDC$, then the triangles OTA, OBT are similar, and OAC, ODB are similar, and hence $OC \cdot OD = OA \cdot OB = OT^2$.

3. If ACB is a right angle and CN perpendicular to AB, then ACB, ANC, BNC are similar triangles, and $CN^2 = AN \cdot NB$, $AC^2 = AN \cdot AB$, $BC^2 = BN \cdot BA$.

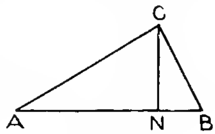

FIG. 13.

4. If three similar figures are described on the sides of a right-angled triangle, the area of that on the hypotenuse equals the sum of the areas on the other sides.

Let K_a, K_b, and K_c be the areas on BC, CA, AB where C is the right angle.

$$\frac{K_a}{K_b} = \left(\frac{CB}{CA}\right)^2.$$

$$\therefore \quad \frac{K_a}{(BC)^2} = \frac{K_b}{(CA)^2} = \text{(similarly)} \frac{K_c}{(AB)^2} = \frac{K_a + K_b}{BC^2 + CA^2} \text{ (p. 18)}$$

$$= \frac{K_a + K_b}{AB^2} \text{ (Eucl. i. 47).}$$

$\therefore$ $K_c = K_a + K_b$.

5. If $ABCD\ldots$, $A'B'C'D'\ldots$ are similar polygons placed in one plane so that AB, $A'B'$ are parallel, then AA', BB', CC', $DD'\ldots$ are concurrent.

[The figures are then said to be *homothetic*, and the point of concurrency is called their *homothetic centre*.]

6. Show that the common tangents to two non-intersecting circles pass through one or other of two points which satisfy the definition of homothetic centres.

Definition. If $ABC...$, $A'B'C'...$, be two similar figures in a plane, and it is possible to find a point O such that

$$\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC} = \ldots,$$

then the figures are said to have a *centre of similitude O*.

7. Show that two circles have an infinite number of centres of similitude lying on the circle described on the line joining their two homothetic centres as diameter.

Projection in one Plane.

Let OX be a fixed line, called the axis of projection.

Let $A, B, C, D, \ldots$ be any points, and let $AK, BL, CM, DN, \ldots$ be perpendiculars on to the axis.

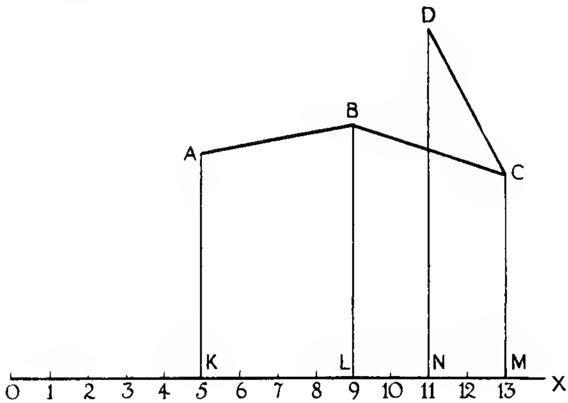

Fig. 14.

Then K, L, M, N are called the projections of A, B, C, D, and the lengths KL, LM, MN the projections of AB, BC, CD on the axis OX.

Now extend the meaning of the symbol AB so as to mean not only the length AB, but the direction also, so that AB means a *displacement* or step from A to B, and BA means a *displacement* from B to A. Then $AB+BA = 0$, not $2\,AB$, and signifies a step from A to B combined with one from B to A.

This extended meaning involves no logical difficulty. It affords a ready means of abbreviating and generalizing many properties, and provides a useful connexion between geometry and algebra. The convention that AB means a displacement may be adopted in any theorem or group of theorems; while the less extended meaning can be used whenever direction is not involved.

Using this convention we have for example that

$$KL + LN + NK = 0,$$

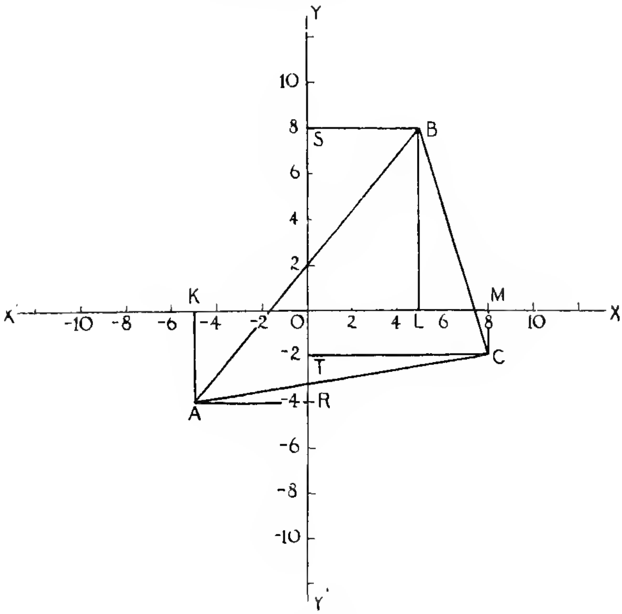

FIG. 15.

where K, L, N are any points in a straight line. In Figure 14 this is equivalent to saying $4 + 2 - 6 = 0$.

Again, $KL = KO + OL = OL - OK$. $(4 = 9 - 5)$

Now if A, B, C are any points in the plane the projection of AB is KL and of BC is LM.

The sum of the projections of AB and BC is

$$KL + LM = KM = \text{projection of } AC;$$

and the sum of the projections of AB, BC, CA is zero.

Similarly, the sum of the projections of BC, CD = the projection of BD; and the sum of the projections of the sides of a closed polygon is zero.

These statements are true for all axes of projection.

Thus in Figure 15, where YOY' is an axis perpendicular to XOX', the projection of AC = sum of projections of AB, BC.

On the axis OX this becomes

$$KM = KL + LM,$$

and on the axis OY,

$$RT = RS + ST.$$

Notice that $OL = +5$, $OM = +8$, $OK = -5$, $OS = +8$, $OT = -2$, $OR = -4$ in the figure,

and $KM = KO + OM = OM - OK = 8 - (-5) = 13,$

$KL = OL - OK = 5 - (-5) = 10,$

$LM = OM - OL = 3.$

Similarly, $RT = OT - OR = -2 - (-4) = 2,$

$RS = OS - OR = 12$, and $ST = OT - OS = -10.$

This method and notation will be used frequently in the sequel.

EXAMPLES.

1. Show that the sum of the projections on a given axis of any broken rectilinear path between two fixed points is constant.

2. C is the middle point of AB, and O any point. Show that on any axis the sum of the projections of OA and OB = twice the projection of OC.

3. Enunciate Euclid ii. 12 and 13 as a single proposition by means of projection.

SECTION III

TRIGONOMETRY

The Trigonometrical Ratios.

I. The Case of a Positive Acute Angle.

Let COD be an acute angle. From Q, any point in OD, draw QM to meet OC at right angles at M (Fig. 16).

Then, if Q_1 is any other position of Q on OD and Q' any

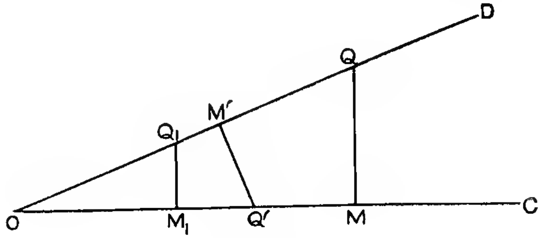

Fig. 16.

position on OC, and $Q_1 M_1$, $Q'M'$ are perpendiculars on OC, OD, the triangles OMQ, $OM_1 Q_1$, $OM'Q'$, are similar.

$$\therefore \quad OM : MQ : QO = OM_1 : M_1 Q_1 : Q_1 O = OM' : M'Q' : Q'O,$$

and any ratio formed by two sides of the triangle OMQ is independent of the position of Q.

The ratio $\dfrac{MQ}{OQ}$ is called the *sine* of the angle COD.

The ratio $\dfrac{MQ}{OM}$ is called the *tangent* of COD.

The ratio $\dfrac{OQ}{OM}$ is called the *secant* of COD.

[The origin of these terms is probably as follows. Using Figure 17, where OP is unity, PP' measures the stretch of a bow-string where

PAP' is the bow; *sinus* in Latin is the fold (of a garment) in one of its uses. (NA was formerly called the *sagitta* or arrow. It is also called the *versed sine* of the angle.) TO is part of a secant to the circle, TA the tangent. The ratios of TO and TA to OA, that is, to unity, are the secant and tangent of COD as just defined.]

In Figure 16, the angle OQM is complementary to COD, and the sine, tangent, and secant of OQM are called the cosine, co-tangent, and co-secant of COD. Then

the ratio $\frac{OM}{OQ}$ is called the *cosine* of COD,

the ratio $\frac{OM}{MQ}$ is called the *co-tangent* of COD,

the ratio $\frac{OQ}{QM}$ is called the *cosecant* of COD.

These six ratios *are the trigonometrical ratios* of the acute angle.

II. All Angles.

These definitions have been extended in accordance with the process of mathematical convention, described on pp. 1, 2, to fit the generalized idea of an angle that follows. This is done with the use of positive and negative lengths measured on two axes as in graphic algebra. The definitions just given are included as particular cases.

The Angle.

Draw a line OA, horizontally from left to right, and regard OA as of unit length. Describe a circle, centre O, radius OA.

Let there be a movable radius, as the hand of a watch. Take OA to be its zero position, and suppose the circumference of the circle to be divided into 360 equal parts; number these parts, as in Figure 17, making the numbers increase in the opposite direction to that in which the hands of an actual watch revolve.

An angle, 'the inclination of two straight lines to one another,' is to be measured as follows: Place one arm, OC, along OA, and let the other arm, OD, intersect the circle at P. Turn the moving radius from OA to OP. The magnitude of the angle is measured by the number of unit divisions of arc through which the radius is turned. The angle subtended by the unit division as numbered in the figure is called a degree. 90 degrees (written 90°) make a right angle.

An angle may be either positive or negative and of any magnitude. In Figure 17 the angles COD, COB, COP_1, COA', COP_2, COB', COP_3 are respectively 35°, 90°, 142°, 180°, 195°, 270°, and 293°. The same angles may equally be read −325°,

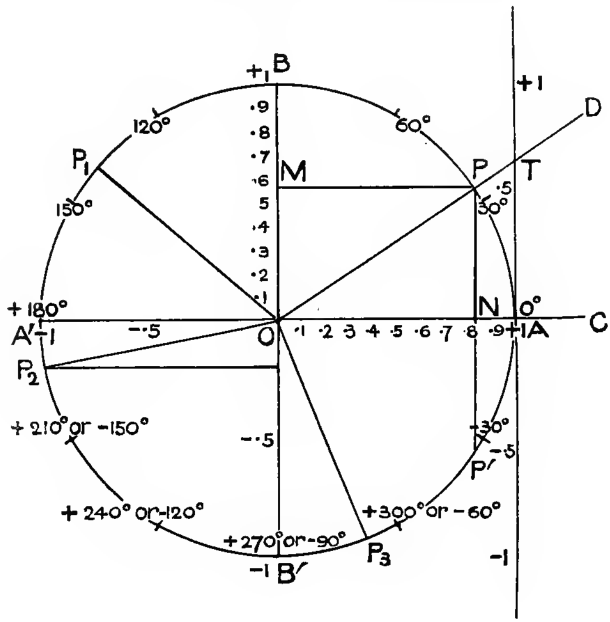

Fig. 17.

−270°, −218°, −180°, −165°, −90°, and −67°. Whether the positive or the negative measurement is to be taken depends on the problem with which the angle is connected. Thus, if we have a revolving wheel with a marked spoke OP which starts from the position OA, it may arrive at OP_1 either by revolving through 142° in a contra-clockwise direction, or through 218° in a clockwise direction. Or it may turn through any number, n, complete revolutions and then through 142° before it stops.

Thus the angle marked as AOP (AOP_1, &c.) may be any one of the angles $n \cdot 360° + x°$, where x is between 0° and +360° and n is zero or any positive or negative integer. If, for example, $n = -8$ and $x = 40$, we should have 8 negative revolutions completed less 40°, and the angle would be reckoned −2840°.

In trigonometrical measurements, it is always to be supposed that the angle is brought to this figure for measurement. This method is closely allied to that ordinarily used in scale drawing, when a protractor is placed on a figure with its zero reading on one of the lines.

Mark scales on the lines AA', BB' from +1 to −1 as in Figure 17.

From every point on the circumference draw perpendiculars to AA' and BB', as PN, PM. Then ON, OM are the projections of OP on OA and OB. The scale measurement of ON is called the *cosine*, and of OM the *sine*, of the angle COD.

Angle x.	Projection of unit length on OA, i.e. cosine x.	Projection of unit length on OB, i.e. sine x.	$\frac{\sin x}{\cos x} = \tan x$.	Angle x.	Projection of unit length on OA, i.e. cosine x.	Projection of unit length on OB, i.e. sine x.	$\frac{\sin x}{\cos x} = \tan x$.
0°	+1.00	0	0	200°	− .94	− .34	+ .36
20°	+ .94	+ .34	+ .36	220°	− .77	− .64	+ .84
40°	+ .77	+ .64	+ .84	240°	− .50	− .87	+1.73
60°	+ .50	+ .87	+1.73	260°	− .17	− .98	+5.67
80°	+ .17	+ .98	+5.67	270°	0	−1.00	∞
90°	0	+1.00	∞	280°	+ .17	− .98	−5.67
100°	− .17	+ .98	−5.67	300°	+ .50	− .87	−1.73
120°	− .50	+ .87	−1.73	320°	+ .77	− .64	− .84
140°	− .77	+ .64	− .84	340°	+ .94	− .34	− .36
160°	− .94	+ .34	− .36	360°	+1.00	0	0
180°	−1.00	+0	0				

The cosine and sine of any angle can readily be measured on a fine drawing to the second decimal place. The table shows the results for angles selected round the circle.*

The ratio of OM to ON is called the *tangent* of the angle. For angles between −90° and +90° the tangent may be easily measured as follows: Let the tangent at A meet OPD at T.

* The student is very strongly recommended to make these measurements for every 10° and to study the relations obtained. There is no better method of obtaining familiarity with these ratios.

Then the triangles ONP, OAT are similar, and we have

$$\text{tangent of } COD \text{ (or } \tan x) = \frac{OM}{ON} = \frac{NP}{ON} = \frac{AT}{OA}.$$

Since OA is of unit length, $\tan x =$ length of AT on the scale shown in Figure 17.

The following are the formal definitions:

The *cosine* of an angle COD is the projection on one arm (OC) of unit length measured on the other (OD).

The *sine* of an angle COD is the projection, on a line (OB) making a positive right angle with OC, of unit length measured on OD.

The *tangent* of an angle is the ratio of the projection on OB to that on OC.

The *cotangent* of an angle is the ratio of the projection on OC to that on OB.

The *secant* of an angle is the ratio of unit length on OD to its projection on OC.

The *cosecant* of an angle is the ratio of unit length on OD to its projection on OB.

It is evident that the same ratios are obtained whatever unit of length is taken, so that the ratios are functions (p. 72 below) of the magnitude of the angle and of nothing else.

We have obviously the following relations:

$$\tan x = \frac{\sin x}{\cos x} = \frac{1}{\cot x}; \quad \sec x = \frac{1}{\cos x}; \quad \operatorname{cosec} x = \frac{1}{\sin x}.$$

sin, cos, tan, cot, sec, and cosec are the abbreviations ordinarily used.

The lengths ON, OM in Figure 17 on p. 41 may have positive or negative values, but in all cases their squares are of course positive, and the equation $ON^2 + NP^2 = OP^2$ is satisfied in the ordinary geometrical sense.

This may be written $\sin^2 x + \cos^2 x = 1$, where $\sin^2 x$ is the abbreviation for $(\text{sine } x)^2$.

By algebraic processes we have

$$\left(\frac{\sin x}{\cos x}\right)^2 + 1 = \left(\frac{1}{\cos x}\right)^2 \text{ and } \therefore \tan^2 x + 1 = \sec^2 x,$$

$$\text{and} \quad 1 + \left(\frac{\cos x}{\sin x}\right)^2 = \left(\frac{1}{\sin x}\right)^2 \text{ and } \therefore 1 + \cot^2 x = \operatorname{cosec}^2 x.$$

It is easily shown by geometry that

(i) when $\angle NOP = 45°$, $ON = OM$, $\therefore \sin x = \cos x$,

$$\therefore\ 2\sin^2 x = 1,\ \sin 45° = \frac{1}{\sqrt{2}} = \cos 45°,$$

it being evident that the positive sign is to be taken.

Also, $\tan 45° = 1 = \cot 45°$.

(ii) when $\angle NOP = 60°$, OPA is an equilateral triangle, and

$$ON = \frac{1}{2} = \cos 60°,\quad \therefore\ NP^2 = 1 - ON^2 = \frac{3}{4},$$

$$\therefore\ OM = NP = \frac{\sqrt{3}}{2} = \sin 60°,\ \therefore\ \tan 60° = \sqrt{3}.$$

(iii) when $\angle NOP = 30°$, $NP = \frac{1}{2} = \sin 30°$,

$$ON = \frac{\sqrt{3}}{2} = \cos 30°,\ \therefore\ \tan 30° = \frac{1}{\sqrt{3}}.$$

In general, the trigonometrical ratios of $x°$, when x is commensurable, are incommensurable; but in the cases just given, and in some others (see pp. 58–61) they are commensurable or can be expressed in terms of simple surds.

Relations between the Ratios of Allied Angles.

The values, as obtained in the table on p. 42, are well exhibited in graphic form, where the number of degrees is measured along a horizontal axis, and the corresponding ratios marked from a vertical scale.

It is evident from Figure 17 that

$$\sin(90° - x°) = \sin(90° + x°)$$

when $x°$ is between 0° and 90°. Hence the graph of the sine from 90° to 0° (EO) is equal and similar to that from 90° to 180° (EF). Also from the same figure

$$\sin(270° - x°) = \sin(270° + x°) = -\sin(90° - x°);$$

hence the graphs from 270° to 180° (GF) and 270° to 360° (GH) are equal and similar to each other and to EO. Also $\sin x$ is positive from 0° to 180°, and negative from 180° to 360°. Hence the sine graph from 0° to 360° consists of four equal and similar parts placed as in Figure 18. As x increases positively or

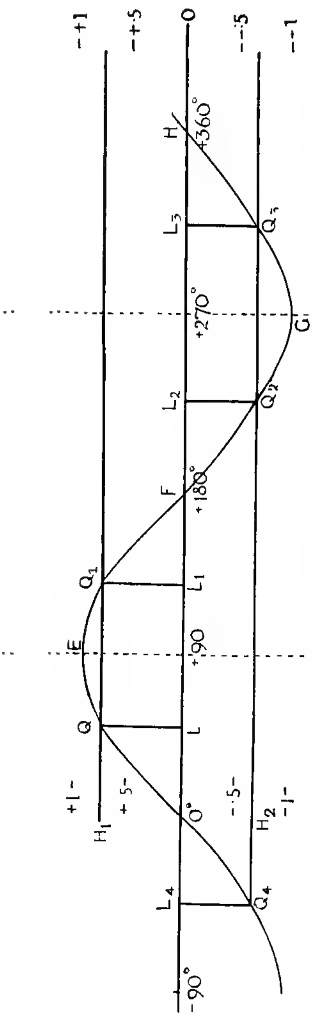

FIG. 18. Graph of sine x. OL shows the number of degrees, LQ, the value of the corresponding sine.

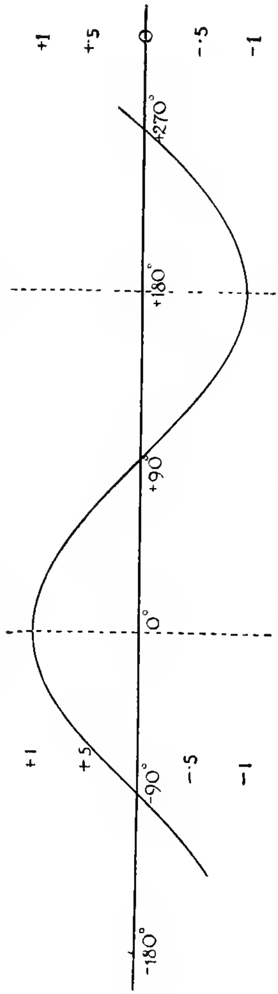

FIG. 19. Graph of cosine x.

negatively without limit this figure is repeated again and again to right and left.

The whole figure, supposed continued indefinitely, is symmetrical about vertical lines through E and through G.

$$\therefore \sin(\overset{L_1Q_1}{90^\circ}+x^\circ) = \sin(\overset{LQ}{90^\circ}-x^\circ) = \sin(n\,.\,360^\circ+90^\circ-x^\circ), \text{ and}$$

$\sin(\overset{L_3Q_3}{270^\circ}+x^\circ) = \sin(\overset{L_2Q_2}{270^\circ}-x^\circ)$ for *all* values of x where n is any positive or negative integer or zero.

If we write x' for $(90-x)$, the first of these equations becomes $\sin(180^\circ-x'^\circ) = \sin x'^\circ = \sin(n\,.\,360^\circ+x'^\circ)$ for all values of x'.

It is now easy to see that the following are true for all values of x.

$$\begin{aligned} \sin \mathbf{x}^\circ &= \sin(180^\circ-\mathbf{x}^\circ) = \sin(\mathbf{n}\,.\,360^\circ+\mathbf{x}^\circ) \\ &= -\sin(-\mathbf{x}^\circ) = -\sin(180^\circ+\mathbf{x}^\circ). \quad . \quad . \quad \text{(i)} \end{aligned}$$

If any angle x° (AOP) is taken, and its cosine ON marked, and at the same time the angle $x^\circ+90^\circ$ (AOP') is taken, and

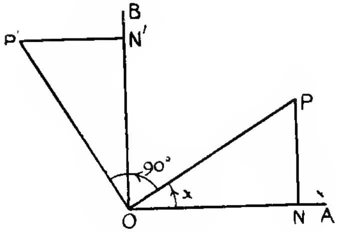

Fig. 20.

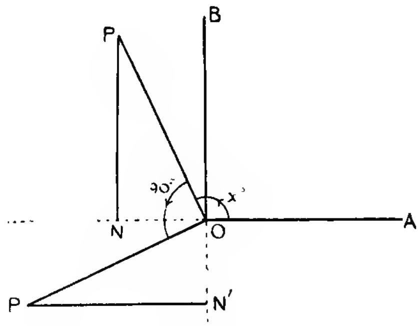

Fig. 21.

its sine ON' marked, as in Figures 20, 21, then it is evident from the definitions of sine and cosine that the figures $ANOP$ and $BN'OP'$ are equal in all respects, that the second is obtained by rotating the first through a positive right angle, whatever the value of x, and that ON and ON' are always the same in sign and equal in magnitude.

Hence $\sin(\mathbf{x}^\circ+90^\circ) = \cos \mathbf{x}^\circ$ for all values of x. . . . (ii)

The cosine graph is therefore obtained by shifting the sine graph to the left through 90° on the horizontal scale (Figure 19).

The symmetry of the cosine graph shows that

$$\cos x^\circ = \cos(-x^\circ) = \cos(n\,.\,360^\circ \pm x^\circ)$$
$$= \sin(90^\circ + x^\circ) \text{ from the previous paragraph } = \sin(90^\circ - x^\circ)$$
$$= -\cos(180^\circ \pm x^\circ).$$

Writing x' for $90 - x$, and then omitting the accent $'$, we have $\cos(90^\circ - x^\circ) = \sin x^\circ$ for all values of x.

Writing $90 + x_1$ for x, in equation (ii) above, and then omitting the suffix $_1$, we have

$$\cos(90^\circ + x^\circ) = \sin(180^\circ + x^\circ) = -\sin x^\circ \text{ from equation (i).}$$

It is then easy to show by division that—

$$\tan x^\circ = \cot(90^\circ - x^\circ), \; \cot x^\circ = \tan(90^\circ - x^\circ)$$
$$\sec x^\circ = \operatorname{cosec}(90^\circ - x^\circ), \; \operatorname{cosec} x^\circ = \sec(90^\circ - x^\circ).$$

[These complementary relations are evident when x is a positive acute angle.]

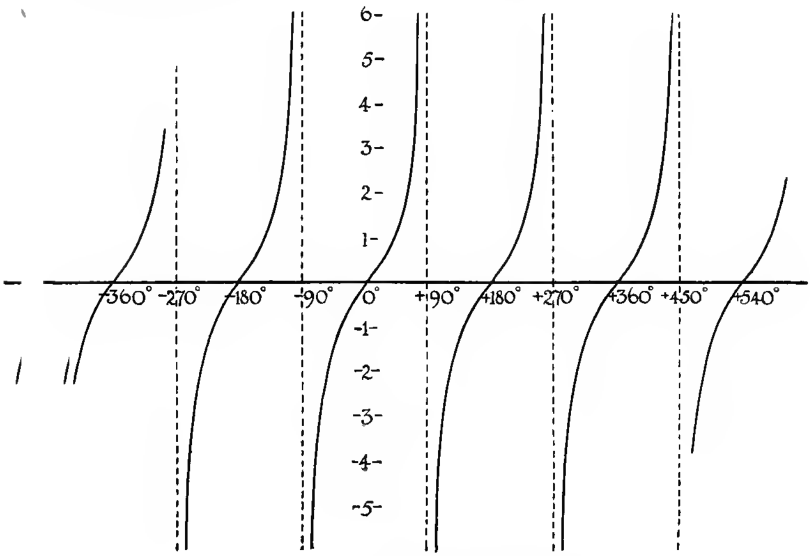

Fig. 22. Graph of tangent x.

We have further—

$$\tan(n\,.\,360^\circ + x^\circ) = \tan x^\circ$$

$$= \frac{\sin x^\circ}{\cos x^\circ} = \frac{-\sin(180^\circ + x^\circ)}{-\cos(180^\circ + x^\circ)} = +\tan(180^\circ + x^\circ)$$

$$= \tan(n\,.\,360^\circ + 180^\circ + x^\circ);$$

$\therefore$ **$\tan \mathbf{x}^\circ = \tan(\mathbf{n}\,.\,180^\circ + \mathbf{x}^\circ)$**, when n is any positive or negative integer, odd or even.

Also $\tan(\mathbf{180^\circ} - \mathbf{x}^\circ) = \dfrac{\sin(180^\circ - x^\circ)}{\cos(180^\circ - x^\circ)} = \dfrac{\sin x^\circ}{-\cos x^\circ} = -\tan \mathbf{x}^\circ.$

These relations can be verified from the graph of $\tan x$, Figure 22.

Projective Methods.

It follows easily from the definitions of sine and cosine that if any length l, CD, is measured on a line whose positive direction makes the angle x with the positive direction of an axis OX,

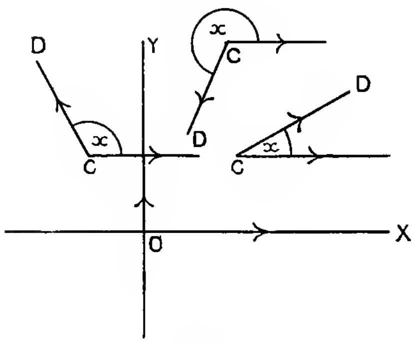

FIG. 23.

the projection of CD on $OX = l\cos x$,

and the projection of CD on $OY = l\sin x$, where OY is an axis making a positive right angle with OX.

This is true for all positions of CD in the plane XOY and is of the greatest importance.

As an instructive example we will obtain some of the relations proved in the last section by projection, using pp. 36–8.

In Figure 24 i, CD makes the angle x° and DC the angle $180^\circ + x^\circ$ with OX.

$\therefore$ $l\cos x^\circ + l\cos(180^\circ + x^\circ) =$ sum of projection of CD, DC on $OX = 0$.

$\therefore$ $\cos x^\circ = -\cos(180^\circ + x^\circ)$.

Similarly, projecting on OY

$$\sin x^\circ = -\sin(180^\circ + x^\circ).$$

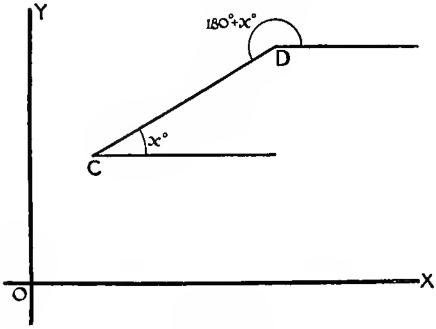

FIG. 24 i.

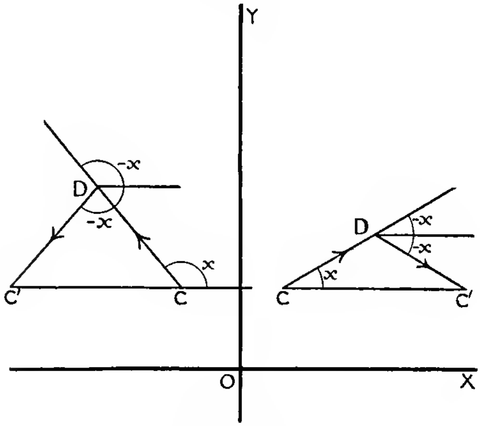

FIG. 24 ii.

Take a step CD in direction x°, turn through a negative angle $2x^\circ$ and take an equal step DC' (Fig. 24 ii). DC' makes $\angle -x^\circ$ with OX. The second step reverses the movement parallel to OY and repeats the movement parallel to OX.

$$\therefore\ l\sin x^\circ + l\sin(-x^\circ) = 0,$$
$$l\cos x^\circ + l\cos(-x^\circ) = 2 \,.\, l\cos x^\circ;$$
$$\therefore\ \sin(-x^\circ) = -\sin x^\circ, \text{ and } \cos(-x^\circ) = \cos x^\circ.$$

Similarly, if we take a step in direction x° and then turn through a positive angle $180^\circ - 2x^\circ$ and take an equal step, we find $\cos x^\circ + \cos(180^\circ - x^\circ) = 0$, and $\cos(180^\circ - x^\circ) = -\cos x^\circ$

$$\sin x^\circ + \sin(180^\circ - x^\circ) = 2\sin x^\circ;$$

$$\therefore \sin(180^\circ - x^\circ) = \sin x^\circ.$$

The relations now obtained are very important, partly because they are needed for using trigonometrical tables, partly because they exhibit the nature of the periodicity of the ratios, that is, the return of the function again and again to the same value as x continually increases.

Using the table on p. 42 from 0° to 90° only, the following examples show how to write down the ratios of any angles.

$$\sin 160^\circ = \sin(180^\circ - 20^\circ) = \sin 20^\circ = +\cdot 34,$$
$$\sin 200^\circ = \sin(180^\circ + 20^\circ) = -\sin 20^\circ = -\cdot 34,$$
$$\sin 320^\circ = \sin(360^\circ - 40^\circ) = -\sin 40^\circ = -\cdot 64,$$
$$\sin 5020^\circ = \sin(13 \times 360^\circ + 340^\circ) = \sin 340^\circ$$
$$= \sin(360^\circ - 20^\circ) = -\sin 20^\circ = -\cdot 34,$$
$$\cos 160^\circ = \cos(180^\circ - 20^\circ) = -\cos 20^\circ = -\cdot 94,$$
$$\cos 200^\circ = \cos(180^\circ + 20^\circ) = -\cos 20^\circ = -\cdot 94,$$
$$\cos 320^\circ = \cos(360^\circ - 40^\circ) = +\cos 40^\circ = +\cdot 77,$$
$$\cos 5020^\circ = \cos(360^\circ - 20^\circ) = \cos 20^\circ = +\cdot 94,$$
$$\tan 160^\circ = -\tan 20^\circ = -\cdot 36, \quad \tan 200^\circ = +\tan 20^\circ = +\cdot 36.$$

Signs of the trigonometrical ratios in the four quadrants.

sin + cos − tan −	sin + cos + tan +
sin − cos − tan +	sin − cos + tan −

It is convenient to use always the relations which involve 180°, rather than those which involve 90°.

Inverse Functions.

In such an equation as $y = \sin x$, suppose that y is given and x is to be found. If $|y|$* > 1, no solution is possible.

Mark the value of y, as OH_1 or OH_2, on the sine graph, p. 45. Draw a horizontal line through H_1 or H_2. This will evidently meet the graph of $\sin x$ at two points between 0° and 360°, two more between 360° and 720° and so on.

Let α° be the smallest positive angle whose sine is y, i.e. OL or OL_2.

* This notation is explained on p. 13.

Then we have two series of angles, viz.

$$\ldots -720^\circ+\alpha^\circ, \quad -360^\circ+\alpha^\circ, \quad \overset{OL \text{ or } OL_2}{\alpha^\circ}, \quad 360+\alpha^\circ, \quad 720^\circ+\alpha^\circ, \ldots$$

and $$\ldots -360^\circ+180^\circ-\alpha^\circ, \quad \overset{OL_1 \text{ or } OL_4}{180^\circ-\alpha^\circ}, \quad 360^\circ+180^\circ-\alpha^\circ,$$
$$720^\circ+180^\circ-\alpha^\circ, \ldots.$$

These are all contained in the formula $n \,.\, 180^\circ+(-1)^n\alpha^\circ$, where n is zero or any positive or negative integer. The formula contains no other angles.

For if we put $n = \ldots -4, -2, 0, 2, 4, \ldots$, we get the first line,
and ,, ,, $\ldots -1, 1, 3, 5, \ldots$, ,, ,, second line.

The equation $y = \sin x$ is in such cases written $x = \sin^{-1}y$ (or in continental usage, $x = \text{arc} \sin y$); these statements are simply abbreviations for 'x is any angle whose sine is y'.

The solution obtained is $x^\circ = n \,.\, 180^\circ+(-1)^n\alpha^\circ$, where α° is an angle between 0° and 360°.

Similar analysis shows that the solutions of $x = \cos^{-1}y$ are

$$x^\circ = n \,.\, 360^\circ \pm \alpha^\circ,$$

and of $x = \tan^{-1}y$ is $x^\circ = n \,.\, 180^\circ+\alpha^\circ$,
where α° is the least positive angle satisfying the equation.

Mensuration involving the ratios of one angle.

Let AB be a chord of circle, radius R, and OA a diameter.

Let $\angle AOB = \alpha^\circ$. Take C, C' on the circle on opposite sides of AB, C being in the greater segment. Cut off unit length, OP, from OA, and draw PN perpendicular to OB.

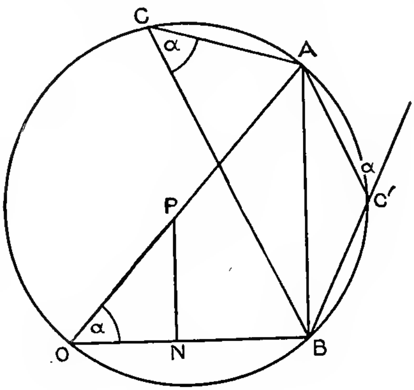

Fig. 25.

Then $\angle ACB = \alpha^\circ$,
$\angle AC'B = 180^\circ-\alpha^\circ$.

$$\sin AC'B = \sin ACB$$
$$= \sin \alpha^\circ = \frac{NP}{OP}$$
$$= \frac{BA}{OA} = \frac{BA}{2R}.$$

$\therefore AB = 2R \sin \alpha^\circ$, where α° is the angle (acute or obtuse) subtended by AB at any point of the circle.

Now consider ABC as any triangle inscribed in the circle. Let A, B, C stand for the numbers of degrees in the angles, and a, b, c for the number of units of lengths of the sides BC, CA, AB.

Then the lengths a, b, c subtend the angles A, B, C, and we have

$$\frac{\mathbf{a}}{\sin \mathbf{A}} = \frac{\mathbf{b}}{\sin \mathbf{B}} = \frac{\mathbf{c}}{\sin \mathbf{C}} = \mathbf{2R}, \quad . \quad . \quad . \quad . \quad \text{(i)}$$

where R is the radius of the circumscribed circle.

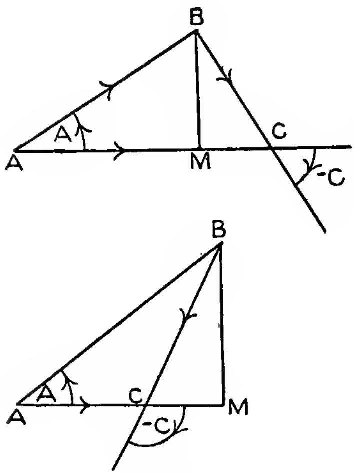

Fig. 26.

Let Δ be the area of the triangle. Choose an acute angle, say A.

Draw BM perpendicular to AC.

Then $\sin A = \dfrac{MB}{AB}$,

$\Delta = \frac{1}{2} AC \times MB$, by elementary mensuration,

$= \frac{1}{2} AC \times AB \sin A$

$= \frac{1}{2} \mathbf{bc} \sin \mathbf{A} = \dfrac{abc}{4R}$

$= \frac{1}{2} \mathbf{ac} \sin \mathbf{B} = \frac{1}{2} \mathbf{ab} \sin \mathbf{C}$

and $\mathbf{R} = \dfrac{\mathbf{abc}}{\mathbf{4\Delta}}$.

Now regard AC as an axis, measured to the right from A. It is easily seen that the directions AB, BC make angles A and $-C$ with AC.

Since the sum of the projections of AB, BC on AC is equal to AC,

$$AB \cos A + BC \cos(-C) = AC;$$

$$\therefore \; c \cos A + a \cos C = b, \text{ since } \cos(-C) = \cos C;$$

$$\therefore \; c \cos A = b - a \cos C.$$

But we have

$$c \sin A = a \sin C.$$

Squaring and adding,

$$c^2(\cos^2 A + \sin^2 A) = b^2 - 2ab\cos C + a^2\cos^2 C + a^2\sin^2 C;$$

$$\left.\begin{aligned} \therefore\quad \mathbf{c}^2 &= \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{ab}\cos \mathbf{C}. \\ \text{Similarly,}\quad \mathbf{a}^2 &= \mathbf{b}^2 + \mathbf{c}^2 - 2\mathbf{bc}\cos \mathbf{A}, \\ \mathbf{b}^2 &= \mathbf{c}^2 + \mathbf{a}^2 - 2\,\mathbf{ca}\cos \mathbf{B}. \end{aligned}\right\} \quad . \; . \; . \quad \text{(ii)}$$

No assumption has been made in these proofs that the angles are acute, and the results are true for all triangles, acute-angled, right-angled, or obtuse-angled.

Solution of triangles, without logarithms, assuming the use of tables of sine and cosine. (Compare p. 30 above and pp. 62–3 below.)

I. Given two sides (a, b) and the contained angle (C).

c is found from the first of equations (ii).

$\cos B$, and hence B, is found from the second. There is only one value of B between 0 and 180 with a given cosine, hence the solution is unique.

Then $A = 180 - B - C$.

II. Given three sides.

$\cos A$, $\cos B$, $\cos C$, and hence A, B, C, are found from equations (ii).

III. Given two angles (and therefore the third, since $A + B + C = 180$) and one side (a).

From equations (i) b and c are found by writing in the value of the sines.

IV. Given two sides (a, b) and an opposite angle A.

From equations (i), $\sin B = \dfrac{b}{a}\sin A$. $b \sin A$ is the altitude of the triangle, if c is taken as the base.

If $a < b\sin A$, there is no solution.

If $a = b\sin A$, $B = 90$, $C = 90 - A$, and $c^2 = a^2 + b^2$.

If $a > b\sin A$, there are two values for B, supplementary to each other, which satisfy the equation. Taking either of these, $C = 180 - A - B$, and c is then obtained from equations (i).

In this case it can be seen from a figure that if $a < b$ both solutions are admissible, and if $a \nless b$ only one solution.

Heights and Distances.

If CD is a vertical post, and A and B two points such that ABC is a straight line and the $\angle ACD$ a right angle, then if AB and the angles DAC, DBC (the 'elevations' of D as seen from A and B) are measured, CD can be found.

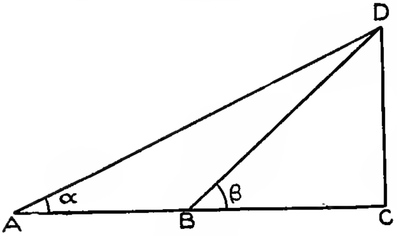

FIG. 27.

Let $\angle DAC = \alpha$, $\angle DBC = \beta$.

Then $AB = AC - BC = CD \cot\alpha - CD \cot\beta$;

$$\therefore\ CD = \frac{AB}{\cot\alpha - \cot\beta}.^*$$

If the elevations of two points C and D in a vertical line are measured, and the perpendicular distance of A from CD ($= AB$) is known, then CD can be found.

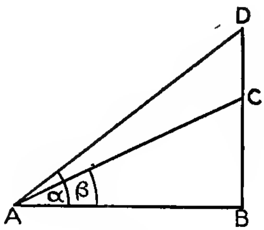

FIG. 28.

For $CD = BD - BC$
$= AB \tan\alpha - AB \tan\beta$.*

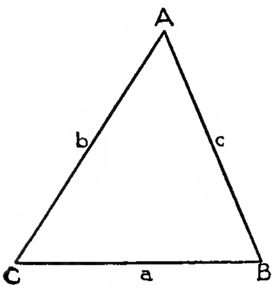

FIG. 29.

If C and B are given points, and length CB is known, then the distance of an object A can be found if the angles ACB, ABC are measured.

This is Case II of the solution of triangles.

When b is found the perpendicular distance of A from BC, which equals $b \sin C$, is known.

* $\therefore\ CD = \dfrac{AB \sin\alpha \sin\beta}{\sin(\beta - \alpha)}$ (see p. 57),

which is adapted to logarithmic computation.

In the next case $CD = \dfrac{AB \sin(\alpha - \beta)}{\cos\alpha \cos\beta}$.

Examples.

1. Show that $\sin^4 x + \cos^4 x = 1 - 2\sin^2 x \, . \cos^2 x$, from the formulae on p. 43.

2. Show that $\sec^2 x + \mathrm{cosec}^2 x = \sec^2 x \, . \mathrm{cosec}^2 x$.

3. Show that $\dfrac{1+\sin x}{\cos x} = \dfrac{\cos x}{1-\sin x}$.

4. Solve the equations

(i) $2\sin x = 1$. (ii) $\sqrt{2} \, . \cos 2x = 1$.

(iii) $\tan 3x = 3$. (iv) $2\cot x = 1$.

[$\tan 3x^\circ = 3 = \tan 72^\circ$ to the nearest degree, from the tables; $\therefore 3x^\circ = n \, . 180^\circ + 72^\circ$, $x^\circ = n \, . 60^\circ + 24^\circ$, $x^\circ = \ldots - 96^\circ$, -36°, 24°, $84^\circ \ldots$.]

Check the solutions by drawing the graphs of $\sin x$, $\cos 2x$, $\tan 3x$, $\cot x$.

5. Find the other angles, and sides, and the area of the triangle, when

(i) $a = 41$, $b = 37$, $c = 18$.

(ii) $A = 27$, $b = 53$, $c = 25$,

(iii) $A = 53$, $B = 100$, $c = 10$.

(iv) $A = 25$, $c = 4$, $a = 3$.

Check the results by drawings to scale.

6. Show by projecting the perimeter of a regular pentagon on to one of its sides that $\cos 72^\circ + \cos 144^\circ + \frac{1}{2} = 0$.

7. The angle of depression of the foot of a vertical cliff at an observer's eye, distant 220 yds. from the cliff, is $2\frac{1}{2}^\circ$ and the angle of elevation of the top is 12°. Show that the cliff is 56·4 yds. high, and that the foot of the cliff is 9·6 yds. below the observer's eye.

8. A passenger in a train travelling along a straight railroad observes that a building whose perpendicular distance from the road is $1\frac{1}{2}$ miles appears to lie about 25°, and, in 48 seconds, about 35° to his right. Show that the train is travelling at about 45 miles per hour.

9. A telegraph post, standing vertically on flat ground, is strengthened by two guy ropes, both fastened to the top of the post and in the same vertical plane. If they are inclined to the horizontal at angles of 47° and 62° and enter the ground 14 ft. apart, show that the post is 35 ft. high, and find the length of each rope.

THE TRIGONOMETRICAL RATIOS OF THE SUM OF TWO ANGLES.

Let $\alpha°$ and $\beta°$ be any two angles, positive or negative.

Let $\alpha°$ be AOP, measured as before from an initial line OA, so that its cosine and sine are the projections of OP on $A'A$, $B'B$, where $ABA'B'$ is a circle of unit radius.

Take OP as the initial line from which to measure $\beta°$, and let POQ be the angle $\beta°$. Q may be anywhere on the circle.

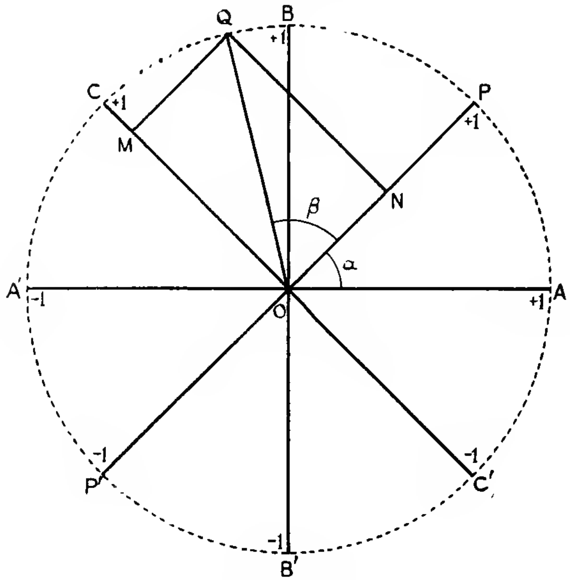

FIG. 30.

Let POC be a positive right angle. Produce PO and CO to meet the circle again at P', C'.

Then the cosine and sine of $\beta°$ are the projections of OQ on $P'P$, $C'C$.

Draw QM, QN perpendicular to OC, OP. M may be anywhere from C' to C, and N anywhere from P' to P.

The projection of OQ on any line equals the sum of the projections of ON, NQ, that is of ON, OM, for all positions (p. 37).

OQ makes $\angle\,\alpha° + \beta°$ with OA; ON, the axis on which $\cos\beta$ is measured, makes $\angle\alpha°$ with OA; and OM, the axis on which $\sin\beta$ is measured, makes $\angle\,(\alpha° + 90°)$ with OA.

$\therefore$ the projections of these lengths on OA are $OQ \cos(\alpha+\beta)$, $ON \cos\alpha$, and $OM \cos(\alpha+90)$, (p. 48);
and the projections of these lengths on OB are $OQ \sin(\alpha+\beta)$, $ON \sin\alpha$, and $OM \sin(\alpha+90)$.

$$\therefore \quad OQ\cos(\alpha+\beta) = ON\cos\alpha + OM\cos(\alpha+90).$$

and $$OQ\sin(\alpha+\beta) = ON\sin\alpha + OM\sin(\alpha+90).$$

But OQ is unity, $ON = \cos\beta$, $OM = \sin\beta$,

$$\cos(\alpha+90) = -\sin\alpha, \quad \text{and} \quad \sin(\alpha+90) = \cos\alpha \text{ (p. 46).}$$

$$\therefore \quad \cos(\alpha+\beta) = \cos\beta\cos\alpha - \sin\beta\sin\alpha$$
$$= \cos\alpha \,.\, \cos\beta - \sin\alpha \,.\, \sin\beta \quad . \quad . \quad \text{(A)}$$

and $$\sin(\alpha+\beta) = \cos\beta\sin\alpha + \sin\beta\cos\alpha$$
$$= \sin\alpha \,.\, \cos\beta + \cos\alpha \,.\, \sin\beta \quad . \quad . \quad \text{(B)}$$

These results are proved for all values of α and β.

If β is negative it is convenient to write $\beta = -\beta'$, where β' is positive.

The first equation becomes

$$\cos(\alpha-\beta') = \cos\alpha\cos(-\beta') - \sin\alpha\sin(-\beta')$$
$$= \cos\alpha\cos\beta' + \sin\alpha\sin\beta'.$$

But since β' is any angle we may just as well write

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta. \quad . \quad . \quad . \quad \text{(C)}$$

Similarly, $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$. . . . (D)

Adding the identities (A) and (C) we have

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta).$$

Subtracting (A) from (C)

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta).$$

Similarly from (B) and (D)

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$
$$2\sin\beta\cos\alpha = \sin(\alpha+\beta) - \sin(\alpha-\beta).$$

Now write γ for $(\alpha+\beta)$ and δ for $(\alpha-\beta)$, so that $\alpha = \frac{1}{2}(\gamma+\delta)$, $\beta = \frac{1}{2}(\gamma-\delta)$.

The equations last written become

$$\left.\begin{aligned} \cos\gamma + \cos\delta &= 2\cos\tfrac{1}{2}(\gamma+\delta)\cos\tfrac{1}{2}(\gamma-\delta) \\ \cos\delta - \cos\gamma &= 2\sin\tfrac{1}{2}(\gamma+\delta)\sin\tfrac{1}{2}(\gamma-\delta) \\ \sin\gamma + \sin\delta &= 2\sin\tfrac{1}{2}(\gamma+\delta)\cos\tfrac{1}{2}(\gamma-\delta) \\ \sin\gamma - \sin\delta &= 2\sin\tfrac{1}{2}(\gamma-\delta)\cos\tfrac{1}{2}(\gamma+\delta) \end{aligned}\right\}$$

These equations are true for all values of γ and δ. They are of use in processes of trigonometrical computation.

Returning to equations (A) and (B), take the particular case where $\beta = \alpha$.

(A) becomes $\cos 2\alpha = \cos\alpha \, . \, \cos\alpha - \sin\alpha \sin\alpha$

$$= \cos^2\alpha - \sin^2\alpha \;\; = 1 - \sin^2\alpha - \sin^2\alpha$$

$$= 1 - 2\sin^2\alpha \;\; = 2\cos^2\alpha - 1 \quad . \quad . \quad . \quad \text{(E)}$$

(B) becomes $\sin 2\alpha = 2\sin\alpha\cos\alpha$ . . . . (F)

Writing x for 2α, (E) becomes

$$\cos x = 1 - 2\sin^2\tfrac{1}{2}x = 2\cos^2\tfrac{1}{2}x - 1\,;$$

$$\left.\begin{aligned} \therefore\ \sin(\tfrac{1}{2}x) &= \pm\sqrt{\tfrac{1}{2}(1-\cos x)}, \\ \cos(\tfrac{1}{2}x) &= \pm\sqrt{\tfrac{1}{2}(1+\cos x)}. \\ \therefore\ \tan(\tfrac{1}{2}x) &= \pm\sqrt{\frac{1-\cos x}{1+\cos x}}. \end{aligned}\right\} \quad . \quad . \quad . \quad \text{(G)}$$

Where x is between $0°$ and $180°$ the ratios of $\frac{1}{2}(x)$ are positive, and the upper signs are to be taken. If we only know $\cos x$ and nothing else as to the value of x both signs are admissible.

From (A) and (B), $\tan(\alpha+\beta) = \dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$.

Divide every term in the fraction by $\cos\alpha \, . \, \cos\beta$, and we have

$$\left.\begin{aligned} \tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\,; \\ \therefore\ \tan 2\alpha &= \frac{2\tan\alpha}{1-\tan^2\alpha}. \\ \text{Similarly,}\quad \tan(\alpha-\beta) &= \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}. \end{aligned}\right\} \quad . \quad . \quad . \quad \text{(H)}$$

As examples of the use of these formulae we will find the ratios of certain angles.

In (A) and (B) let $\alpha = 45°$, $\beta = 30°$,

$$\cos 75° = \cos(45° + 30°) = \cos 45° \cos 30° - \sin 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \sin 15°,$$

since $15°$ and $75°$ are complementary.

Similarly, $\sin 75^\circ = \frac{1}{\sqrt{2}} \cdot \frac{1}{2} + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \cos 15^\circ.$

From (H) $\tan 75^\circ = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 \times \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

$$= \frac{(\sqrt{3}+1)^2}{(\sqrt{3})^2 - 1} = \frac{3 + 1 + 2\sqrt{3}}{3 - 1} = 2 + \sqrt{3} = \cot 15^\circ;$$

$$\tan 15^\circ = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} = 2 - \sqrt{3}.$$

Also $\cot 15^\circ = \tan 75^\circ = 2 + \sqrt{3}$, and $\cot 75^\circ = 2 - \sqrt{3}$.

In (G) take $x = 45^\circ$.

$$\sin 22\tfrac{1}{2}^\circ = \sqrt{\tfrac{1}{2}(1 - \cos 45^\circ)} = \sqrt{\tfrac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)}$$
$$= \tfrac{1}{2}\sqrt{2 - \sqrt{2}} = \cos 67\tfrac{1}{2}^\circ,$$

$$\cos 22\tfrac{1}{2}^\circ = \sqrt{\tfrac{1}{2}(1 + \cos 45^\circ)} = \tfrac{1}{2}\sqrt{2 + \sqrt{2}} = \sin 67\tfrac{1}{2}^\circ,$$

$$\tan 22\tfrac{1}{2}^\circ = \sqrt{\frac{1 - \frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}}} = \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} = \sqrt{\left\{\frac{(\sqrt{2}-1)^2}{(\sqrt{2})^2 - 1}\right\}}$$
$$= \sqrt{\frac{(\sqrt{2}-1)^2}{2-1}} = \sqrt{2} - 1 = \cot 67\tfrac{1}{2},$$

$$\cot 22\tfrac{1}{2} = \sqrt{2} + 1 = \tan 67\tfrac{1}{2}.$$

We have now (see p. 44) found the ratios of 0°, 15°, $22\tfrac{1}{2}$°, 30°, 45°, 60°, $67\tfrac{1}{2}$°, 75°, and 90°. The student is advised to work these out numerically, to make a table of them, and to compare them with the graphs on p. 45.

Of the great number of formulae that can be based on those now given, the following are of special importance:

$$\sin x = 2 \sin \tfrac{1}{2}x \cdot \cos \tfrac{1}{2}x \text{ (from F)} = 2 \cdot \frac{\sin \frac{1}{2}x}{\cos \frac{1}{2}x} \cdot \cos^2 \tfrac{1}{2}x$$
$$= \frac{2 \tan \frac{1}{2}x}{\sec^2 \frac{1}{2}x} = \frac{2 \tan \frac{1}{2}x}{1 + \tan^2 \frac{1}{2}x};$$

$$\cos x = \cos^2 \tfrac{1}{2}x - \sin^2 \tfrac{1}{2}x \text{ (from E)} = \cos^2 \tfrac{1}{2}x (1 - \tan^2 \tfrac{1}{2}x)$$
$$= \frac{1 - \tan^2 \frac{1}{2}x}{\sec^2 \frac{1}{2}x} = \frac{1 - \tan^2 \frac{1}{2}x}{1 + \tan^2 \frac{1}{2}x};$$

$$\tan x = \frac{2 \tan \frac{1}{2}x}{1 - \tan^2 \frac{1}{2}x}.$$

Hence all the ratios of an angle, x, can be expressed rationally in terms of one quantity, viz. $\tan \frac{1}{2}x$.

$$\begin{aligned}\operatorname{Sin}(\alpha+\beta)\times\sin(\alpha-\beta) &= (\sin\alpha\cos\beta)^2-(\cos\alpha\sin\beta)^2,\\ &= \sin^2\alpha\,(1-\sin^2\beta)-(1-\sin^2\alpha)\sin^2\beta,\\ &= \sin^2\alpha-\sin^2\beta = \cos^2\beta-\cos^2\alpha.\end{aligned}$$

Put $\beta = 2\alpha$ in (A), (B), and (H).

$$\begin{aligned}\cos 3\alpha &= \cos\alpha\cos 2\alpha-\sin\alpha\,.\,\sin 2\alpha\\ &= \cos\alpha\,(2\cos^2\alpha-1)-\sin\alpha\,.\,2\sin\alpha\cos\alpha \text{ from (E) and (F)},\\ &= 2\cos^3\alpha-\cos\alpha-2\cos\alpha\,(1-\cos^2\alpha) \text{ since } \sin^2\alpha = 1-\cos^2\alpha,\\ &= 4\cos^3\alpha-3\cos\alpha.\end{aligned}$$

Similarly $\sin 3\alpha = 3\sin\alpha-4\sin^3\alpha$

$$\tan 3\alpha = \frac{\tan\alpha+\tan 2\alpha}{1-\tan\alpha\tan 2\alpha} = \frac{\tan\alpha+\dfrac{2\tan\alpha}{1-\tan^2\alpha}}{1-\tan\alpha\dfrac{2\tan\alpha}{1-\tan^2\alpha}}$$

$$= \frac{(1-\tan^2\alpha)\tan\alpha+2\tan\alpha}{1-\tan^2\alpha-2\tan^2\alpha} = \frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}.$$

As a general example of the preceding methods and formulae we will solve the equation $\cos 3x^\circ = \sin 2x^\circ$.

We have $4\cos^3 x-3\cos x = 2\sin x\cos x$.

$\therefore$ *either* $\cos x = 0$, and $x^\circ = n\,.\,360^\circ \pm 90^\circ$ (p. 51),

or, $4\cos^2 x-3 = 2\sin x$.

$\therefore$ $4(1-\sin^2 x)-3 = 2\sin x$.

$\therefore$ $4\sin^2 x+2\sin x-1 = 0$.

Solving this as a quadratic in $\sin x$, we have

$$\sin x = \frac{-2\pm\sqrt{(2^2+4\times 4)}}{2\times 4} = \frac{-1\pm\sqrt{5}}{4}.$$

Returning to the first equation, we have

$$\cos 3x^\circ = \sin 2x^\circ = \cos(90^\circ-2x^\circ).$$

$\therefore$ $3x^\circ = n\,.\,360^\circ \pm (90^\circ-2x^\circ)$. (p. 51.)

Taking the lower sign, $x^\circ = n\,.\,360^\circ-90^\circ$.

Taking the upper sign, $5x^\circ = n\,.\,360^\circ+90^\circ$,

$$x^\circ = n\,.\,72^\circ+18^\circ.$$

If $n = 0$, $x^\circ = 18^\circ$; if $n = 1$, $x^\circ = 90^\circ$; if $n = -1$, $x^\circ = -54^\circ$.

Hence the possible values of $x°$ from $-90°$ to $+90°$ are $-54°$, $18°$, and $90°$.

Now $\sin 18°$ is +ve, $\sin(-54°)$ is −ve, $\sin 90°$ is 1.

Hence $\dfrac{-1+\sqrt{5}}{4}$ is $\sin 18°$

and $\dfrac{-1-\sqrt{5}}{4}$ is $\sin(-54°)$.

$$\therefore \frac{1+\sqrt{5}}{4} = \sin 54° = \cos 36°,$$

$$\frac{-1+\sqrt{5}}{4} = \sin 18° = \cos 72°.$$

The other ratios of these angles are more complicated in expression.

The following method is of frequent application.

To express $a \cos x + b \sin x$ *in one term*, where a and b are any real quantities,

Write $\dfrac{a}{\sqrt{a^2+b^2}} = \cos \alpha$,

$$\frac{b}{\sqrt{a^2+b^2}} = \sin \alpha.$$

This can always be done without inconsistency.

Then $a \cos x + b \sin x$

$$= \sqrt{a^2+b^2} \,.\, \cos \alpha \cos x + \sqrt{a^2+b^2} \,.\, \sin \alpha \sin x$$

$$= \sqrt{a^2+b^2} \cos (x - \alpha).$$

E.g. $y = 3 \cos x + 2 \sin x$

$= \sqrt{13} \cos (x - \alpha)$, where $\alpha° = \tan^{-1} \frac{2}{3} = 33\frac{1}{2}°$ approx.

$\therefore y = 3{\cdot}61 \cos(x - 33\frac{1}{2}°)$ app.

In Figure 31 the dotted curves represent $3 \cos x$ and $2 \cos x$, and the full curve the algebraic sum.

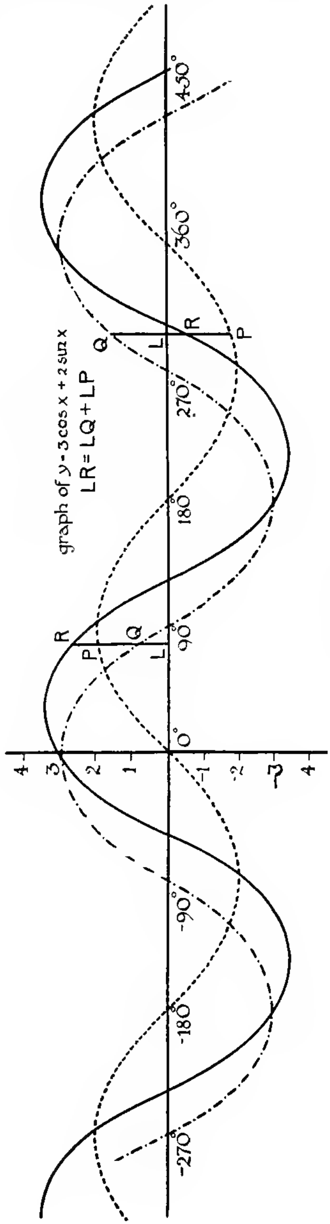

FIG. 31.

The greatest and least values of y are $\pm 3{\cdot}61$, when $x^\circ = 33\frac{1}{2}^\circ$, $213\frac{1}{2}^\circ \ldots$.

Also $y = 0$, when $\cos(x^\circ - 53\frac{1}{4}^\circ) = 0 = \cos 90^\circ$,

Then $x = 33\frac{1}{2} + n\,360 \pm 90 = -56\frac{1}{2},\ 123\frac{1}{2},\ 303\frac{1}{2}$, &c.

Applications to the Sides and Angles of a Triangle, and Solution of Triangles.

Let $2s = a + b + c$. Then $2(s-a) = b + c - a$, $2(s-b) = a - b + c$, $2(s-c) = a + b = c$.

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \text{ (p. 53), and, by p. 58,}$$

$$2\cos^2\frac{C}{2} = 1 + \cos C = \frac{2ab + a^2 + b^2 - c^2}{2ab} = \frac{(a+b+c)(a+b-c)}{2ab};$$

$$\therefore\ \cos\tfrac{1}{2}C = \sqrt{\frac{s(s-c)}{ab}};$$

$$2\sin^2\frac{C}{2} = 1 - \cos C = \frac{2ab - a^2 - b^2 + c^2}{2ab}$$

$$= \frac{(a-b+c)(-a+b+c)}{2ab};$$

$$\therefore\ \sin\tfrac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{ab}};$$

$$\sin C = 2\sin\tfrac{1}{2}C\cos\tfrac{1}{2}C = 2\sqrt{\frac{s(s-a)(s-b)(s-c)}{ab}}.$$

The area, Δ, $= \frac{1}{2}ab\sin C = \sqrt{s(s-a)(s-b)(s-c)}$;

$$\tan\tfrac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{(s-a)(s-b)}{\Delta}.$$

If a, b, c are given, the angles can at once be found, and hence we can deal with Case II (p. 53) of the solution of triangles.

$$\text{Again,}\quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin A + \sin B}{a+b} = \frac{\sin A - \sin B}{a-b}$$

(see pp. 52 and 17);

$$\therefore\ \frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{2\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B)}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)} \text{ (p. 57)}$$

$$= \tan\tfrac{1}{2}(A-B)\cot\tfrac{1}{2}(A+B)$$

$$= \tan\tfrac{1}{2}(A-B)\tan\tfrac{1}{2}C, \text{ since } \tfrac{1}{2}(A+B) \text{ and } \tfrac{1}{2}C \text{ are}$$

complementary.

If a, b, c are given (Case I, p. 53, o fthe solution of triangles), $\frac{1}{2}(A-B)$ is found from the last equation, and $\frac{1}{2}(A+B) = 90 - \frac{1}{2}C$ is known.

Hence A and B are found, and $c = a \cdot \dfrac{\sin C}{\sin A}$.

By these means we can dispense with the direct use of equations (ii) of p. 53, and the whole work depends on factors for which logarithmic tables can be used.

Examples.

1. Find the trigonometrical ratios of 3°, given those of 15° and 18° (pp. 57, 58, and 61).

2. Hence find the ratios for 33°.

3. Find the ratios of $7\frac{1}{2}°$.

4. Solve the equation $4\cos x - 3\sin x = 5$.

5. Show that the radii of the inscribed and escribed circles of a triangle are $\dfrac{\Delta}{s}$, $\dfrac{\Delta}{s-a}$, $\dfrac{\Delta}{s-b}$, $\dfrac{\Delta}{s-c}$ in the notation of p. 62.

6. Show that $\Delta = \sqrt{(r \cdot r_1 \cdot r_2 \cdot r_3)}$ where the letters stand for the radii of the inscribed and escribed circles.

7. Find the angles and the area of the triangle whose sides are 7, 6, 4 inches, by use of the formulae of p. 62. Check the result by a drawing to scale.

8. Find the other angles of a triangle whose sides of 8 and 10 inches include an angle 25°.

9. If A, B, and C are the angles of a triangle, show that

$$\sin A + \sin B + \sin C = 4\cos\tfrac{1}{2}A \cdot \cos\tfrac{1}{2}B \cdot \cos\tfrac{1}{2}C.$$

(Use formulae, p. 57, and (F)).

The Circle and Circular Measure.

Let a regular polygon, ABC ..., of n sides be inscribed in a circle, radius r, centre O.

At the angular points A, B, C, ... draw tangents so as to make another regular polygon, RST ..., circumscribed about the circle.

The angles at the centre, AOB, BOC, &c., are each $\dfrac{360°}{n}$.

OR, *OS*, ... bisect these angles. The angles *AOR*, *ROB*, *BOS*, &c., are each $\frac{180°}{n}$.

OR meets *AB* at right angles, at *M*, and bisects it.

Then $AB = 2\,AM = 2\,AO \sin \frac{180°}{n} = 2\,r \sin \frac{180°}{n}$.

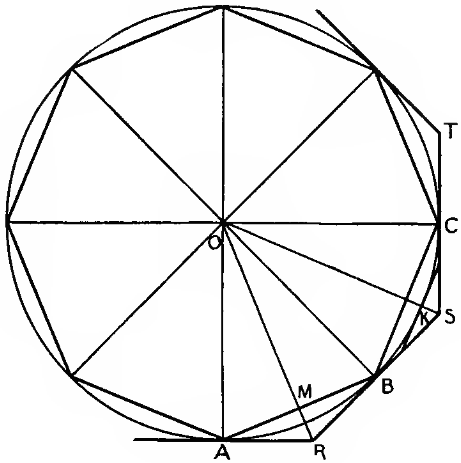

FIG. 32.

The perimeter of the inscribed polygon

$$= n\,.\,AB = 2r\,.\,n \sin \frac{180°}{n}.$$

Area $AOB = AM\,.\,MO = r \sin \frac{180°}{n} \cdot r \cos \frac{180°}{n}$.

The area of the inscribed polygon

$$= \text{Area } AOB \times n = r^2.\, n \sin \frac{180°}{n} \cos \frac{180°}{n} = \tfrac{1}{2} r^2.\, n \sin \frac{360°}{n}.$$

$RS = 2\,RB = 2\,.\,OB \tan ROB = 2\,r\,.\,\tan \frac{180°}{n}$.

The perimeter of the circumscribed polygon

$$= n\,.\,RS = 2\,r\,.\,n \tan \frac{180°}{n}.$$

$$\text{Area } ROS = OB\,.\,RB = r\,.\,r\tan\frac{180^\circ}{n}.$$

The area of the circumscribed polygon

$$= n\,.\,ROS = r^2.\,n\tan\frac{180^\circ}{n}.$$

From pp. 58–9 we can calculate the trigonometrical ratios of

$$\frac{180^\circ}{2^3},\ \frac{180^\circ}{2^4},\ \frac{180^\circ}{2^5}*, \ldots,$$

successively. The work becomes very arduous as the index increases, but presents no difficulty. Hence we can construct the following table :

Perimeter of Polygon.

No. of sides.	[A] Inscribed. $2r\times$	[B] Circumscribed. $2r\times$
4	$4\sin 45^\circ = 2\sqrt{2} = 2{\cdot}83\ldots$	$4\tan 45^\circ = 4.$
8	$8\sin 22\tfrac{1}{2}^\circ = 8\times{\cdot}383 = 3{\cdot}06$	$8\tan 22\tfrac{1}{2}^\circ = 8\times{\cdot}414 = 3{\cdot}313\ldots$
16	$16\sin 11\tfrac{3}{4}^\circ = 16\times{\cdot}195 = 3{\cdot}12\ldots$	$16\tan 11\tfrac{1}{4}^\circ = 16\times{\cdot}1989 = 3{\cdot}183\ldots$
32	$32\sin 5\tfrac{5}{8}^\circ = 32\times{\cdot}09803 = 3{\cdot}13635$	$32\tan 5\tfrac{5}{8}^\circ = 32\times{\cdot}09849 = 3{\cdot}1517\ldots$
64	$64\sin 2\tfrac{13}{16}^\circ = 64\times{\cdot}04907 = 3{\cdot}1405\ldots$	$64\tan 2\tfrac{13}{16}^\circ = 64\times{\cdot}049127 = 3{\cdot}1441\ldots$

Area of Polygon.

No. of sides.	Inscribed. $r^2\times$	Circumscribed. $r^2\times$
4	$2\sin 90^\circ = 2.$	4.
8	$4\sin 45^\circ = 2{\cdot}83$	3·313
16	$8\sin 22\tfrac{1}{2}^\circ = 3{\cdot}06$	3·183
32	$16\sin 11\tfrac{3}{4}^\circ = 3{\cdot}12$	3·1517...
64	$32\sin 5\tfrac{5}{8}^\circ = 3{\cdot}136$	3·1443...

By elementary geometry the circumscribed polygon has greater area and perimeter than has the inscribed polygon of the same number of sides. $\therefore\ \sin\frac{180^\circ}{2^m} < \tan\frac{180^\circ}{2^m}.$

Also, each time the number of sides is doubled the area and

* $\cos\frac{180}{2^3} = \frac{1}{2}\sqrt{2+\sqrt{2}},\quad \cos\frac{180}{2^4} = \sqrt{\frac{1+\cos\frac{180}{2^3}}{2}} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}},$

$\cos\frac{180}{2^5} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$, and $\sin\frac{180}{2^5} = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$, &c.

perimeter of the inscribed polygon are increased, while those of the circumscribed polygon are diminished.

Hence

$$2^{m-1}\sin\frac{180^\circ}{2^{m-1}} < 2^m \sin\frac{180^\circ}{2^m} < 2^m \tan\frac{180^\circ}{2^m} < 2^{m-1}\tan\frac{180^\circ}{2^{m-1}}.$$

Thus the numbers in column [A] must continually increase as we read down, while those in [B] decrease; but however far we continue the table, every number in column [A] must always be less than any number in column [B].

In the few cases taken, the final numbers only differ in the 3rd decimal place.

If we had taken 2^{12} sides the difference between the last numbers in [A] and [B] would have been $1 \div 10^6$ nearly.

Hence $n \sin\frac{180}{n}$ and $n \tan\frac{180}{n}$ tend towards the same value as n is increased by continual doubling, and this value is between 3·1405 and 3·1441.

Definitions. The limit towards which $n \sin\frac{180}{n}$ tends as n is increased indefinitely* is called π. In the notation of p. 100 $\underset{n \to \infty}{\text{Lt}}\ n \sin\frac{180}{n} = \pi$.

π has been evaluated with great accuracy by this and other processes. Its value is 3·14159....

The limits towards which the perimeter and area of a regular† inscribed polygon tend, as the number of its sides is indefinitely increased, are defined to be the measures of the circumference and area of the circle.

[This apparently cumbersome definition is necessary, as soon as we leave pure geometry, principally because no commensurable line or area can be found to measure the circumference and area of a circle, or indeed of any common curve.]

* Here we increase n by continual doubling. It is beyond our scope to show that the same limit is obtained by other steps of increase.

† Regularity can be shown to be unnecessary.

Hence the measure of the **circumference** of a circle, radius r, is $\mathbf{2r.\pi}$; and the measure of the area is

$$r^2 \underset{n\to\infty}{\mathrm{L}^t} \frac{n}{2}\sin\frac{360}{n} = r^2 \underset{m\to\infty}{\mathrm{L}^t} m\sin\frac{180}{m},$$

where $m = \frac{1}{2}n$, $= \pi r^2$.

It is easily shown by the laws of limits (p. 104) that

$$\underset{n\to\infty}{\mathrm{L}^t} n\tan\frac{180}{n} = \underset{n\to\infty}{\mathrm{L}^t} n\sin\frac{180}{n}\sec\frac{180}{n}$$

$$= \underset{n\to\infty}{\mathrm{L}^t} n\sin\frac{180}{n} \times \underset{n\to\infty}{\mathrm{L}^t} \sec\frac{180}{n} = \pi\times 1 = \pi.$$

Hence the perimeter and area of the circumscribed polygon also tend towards $2\pi r$ and πr^2 as the number of sides is increased.

The *length* of an *arc* of a circle, such as AP, is obtained thus:

When n is greatly increased P will lie between two near angular points P_1, P_2 of the polygon or at one of them. By increasing n, P_1 may be made to approach P_2, and either may be substituted for P with as small an error as we can assign. If t sides occupy the part AP_1, then the corresponding part of the perimeter of the polygon is $\frac{t}{n}$ of its complete perimeter. The length of the arc is then defined as $\frac{t}{n}$ of $2\pi r$, when n (and therefore t) is increased indefinitely. The *area* of the *sector* AOP is similarly defined as $\frac{t}{n}$ of πr^2, and $\therefore$ = arc $AP \times \frac{1}{2}r$.

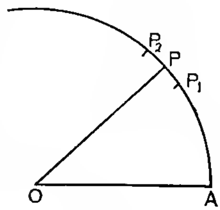

FIG. 33.

The ratio of the length of the arc AP to the radius OA is called the *circular* or *radian measure* of the angle AOP. It is easily seen that this ratio is independent of the size of the figure. It is most conveniently measured on a circle of unit radius. The circular measure of the angle AOP is t times that of an angle standing on a single side of the polygon just described and is $\frac{t}{n}$ of the circular measure of four right angles. Equal angles have equal circular measures, and the ratio of the circular measures of two angles is the same as the ratio of the measurements by degrees, &c., at the beginning of this section.

A *radian* is that angle whose circular measure is unity, i.e. the angle subtended by an arc equal to the radius. The radian is the unit of circular measurement.

The circular measure of four right angles is 2π, being the circumference of the circle whose radius is unity.

$\therefore$ 2π radians = 360 degrees; 1 radian $= \frac{360}{2\pi} = 57 \cdot 3$ degrees approx.

In general, x degrees $= \frac{1}{180}\pi x$ radians.

An angle can thus be converted easily from degrees to radians and vice versa, e.g. $90^\circ = \frac{1}{2}\pi$ radians, $45^\circ = \frac{1}{4}\pi$ radians, $1^\circ = \frac{1}{180}\pi$ $= \cdot 0174533\ldots$ radians, &c.

From the definition, if θ is the circular measure of an angle AOP,

$$\theta = \frac{\text{arc } AP}{OA};$$

$$\textbf{arc AP} = \boldsymbol{\theta} \times \mathbf{r}, \text{ where } r = OA,$$

and $\quad$ **area of sector AOP** $= \textbf{arc AP} \times \frac{1}{2}\mathbf{r} = \frac{1}{2}\mathbf{r}^2\boldsymbol{\theta}.$

$\sin \frac{1}{3}\pi$ is used as the abbreviation for the sine of $\frac{1}{3}\pi$ radians and is the same as $\sin 60^\circ$, and similarly with all the ratios.

The following is a type of simple problems involving circular measure:

At what radius does an arc 10 feet long subtend 15°?

$$15^\circ = \tfrac{1}{12}\pi \text{ radians.}$$

$\therefore$ $10 = \frac{1}{12}\pi \times r$, when r feet is the radius;

$\therefore$ $r = \frac{120}{\pi} = 38 \cdot 2 \ldots.$

Note that $\frac{1}{\pi} = \cdot 31830\ldots.$

Approximate values of the ratios of small angles.

AOP is a sector of radius r, and *OP* meets the tangent at *A* in *T*, and $\angle AOP = \theta$ radians.

From the method of definition of an arc, the arc *AP* lies throughout its length between the chord *AP* and the tangent *AT*.

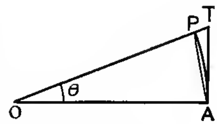

FIG. 34.

$\therefore$ Area $TOA >$ sector AOP

$> $ triangle AOP.

$$\therefore \tfrac{1}{2}r \,.\, r \tan\theta > \tfrac{1}{2}r^2 .\, \theta > \tfrac{1}{2}r^2 \sin\theta \text{ (pp. 65 and 67).}$$

$$\therefore \quad \tan\theta > \theta > \sin\theta. \qquad \text{(i)}$$

This applies to all positive values of $\theta < \frac{1}{2}\pi$.

The following table, illustrated by Figure 35, shows some of the numerical values:

Degrees.	Sine.	Radian measure.	Tangent.
0	0	0	0
1	·0174524	·0174533	·017455
2	·034900	·034907	·03492
3	·05234	·05236	·05241
4	·06976	·06981	·06993
5	·08716	·08727	·08749
10	·17365	·17453	·17633
20	·34202	·34907	·36397
30	·50000	·52360	·57735

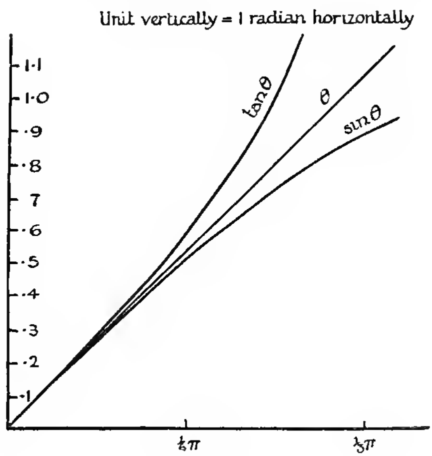

FIG. 35.

When θ is a positive acute angle,

$$\sin\theta = 2\sin\tfrac{1}{2}\theta \,.\, \cos\tfrac{1}{2}\theta = 2\tan\tfrac{1}{2}\theta \,.\, \cos^2\tfrac{1}{2}\theta$$
$$= 2\tan\tfrac{1}{2}\theta \,.\, (1-\sin^2\tfrac{1}{2}\theta) \quad \text{(p. 43)}$$
$$> 2 \,.\, \tfrac{1}{2}\theta\,\{1-(\tfrac{1}{2}\theta)^2\},$$

since $\tan\frac{1}{2}\theta > \frac{1}{2}\theta$ and $\sin\frac{1}{2}\theta < \frac{1}{2}\theta$ from (i).

$$\therefore \quad \theta > \sin\theta > \theta - \tfrac{1}{4}\theta^3 \quad \text{(ii)}$$

$$\cos\theta = 1 - 2\sin^2\tfrac{1}{2}\theta \quad \text{(p. 58)}$$
$$> 1 - 2\,(\tfrac{1}{2}\theta)^2;$$

$$\therefore \quad 1 > \cos\theta > 1 - \tfrac{1}{2}\theta^2, \quad \text{(iii)}$$

and $\cos\theta = 1 - 2\sin^2\frac{1}{2}\theta < 1 - 2\,\{(\frac{1}{2}\theta) - \frac{1}{4}(\frac{1}{2}\theta)^3\}^2$ from (ii),

$$< 1 - \tfrac{1}{2}\theta^2 + \tfrac{1}{16}\theta^4 - \tfrac{1}{512}\theta^6;$$

$$\therefore \; 1 - \tfrac{1}{2}\theta^2 < \cos\theta < 1 - \tfrac{1}{2}\theta^2 + \tfrac{1}{16}\theta^4. \quad \text{(iv)}$$

These relations may be thus expressed

$$\sin\theta = \theta - \kappa_1 \,.\, \tfrac{1}{4}\theta^3,$$
$$\cos\theta = 1 - \tfrac{1}{2}\theta^2 + \kappa_2 \,.\, \tfrac{1}{16}\theta^4,$$

where $\kappa_1\,\kappa_2$ are unknown positive proper fractions depending on θ.

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} < \dfrac{\theta}{1-\frac{1}{2}\theta^2}$, for the numerator of the second fraction is greater and the denominator less than that of the first;

and similarly $\qquad \tan\theta > \dfrac{\theta - \frac{1}{4}\theta^3}{1 - \frac{1}{2}\theta^2 + \frac{1}{16}\theta^4}$.

These formulae are developed further on p. 241.

The following result is used on p. 99.

$$\tan\theta - \sin\theta = \tan\theta\,(1-\cos\theta)$$
$$= 2\tan\theta \,.\, \sin^2\tfrac{1}{2}\theta < 2 \,.\, \frac{\theta}{1-\frac{1}{2}\theta^2} \,.\, (\tfrac{1}{2}\theta)^2 < \frac{\theta^3}{2-\theta^2};$$

$\therefore \; \dfrac{1}{\theta}(\tan\theta - \sin\theta) < \dfrac{1}{\dfrac{2}{\theta^2} - 1}$, where θ is the radian measure of an acute angle.

Examples.

1. $\sin 1^\circ = \frac{1}{180}\pi - \frac{1}{4}\kappa_1\,(\frac{1}{180}\pi)^3$

$$= \cdot 0174533 - \kappa_1 \times \cdot 0000013;$$

$\therefore$ $\sin 1^\circ$ is between $\cdot 0174533$ and $\cdot 0174520$.

2. $\frac{1}{180}\pi < \tan 1° < \dfrac{\cdot 0174533}{1-\frac{1}{2}\left(\frac{\pi}{180}\right)^2} < \cdot 017456,$

tan 1° is between ·0174533 and ·017455.

3. $\cos 10° > 1 - \frac{1}{2}\left(\frac{1}{18}\pi\right)^2 > 1 - \cdot 015233 > \cdot 984767,$

$< 1 - \cdot 015233 + \cdot 000058 < \cdot 984825.$

4. The distance of C, the centre of a circular disc which subtends an angle θ at a point O, is given by

$$d \sin \tfrac{1}{2}\theta = r.$$

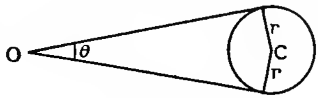

Fig. 36.

If θ is less than 1°, $\dfrac{r}{d}$ differs from $\frac{1}{2}\theta$ by less than $\frac{1}{4}\left(\frac{\pi}{360}\right)^3$, i.e. by less than one part in five millions.

In such cases the radian measure may be used as a very close approximation to the sine or tangent.

SECTION IV

EXPLICIT FUNCTIONS. GRAPHIC REPRESENTATION. EQUATIONS

We have already used the idea of functions, and have represented the functions 10^x, $\log x$, $\sin x$, $\cos x$, and $\tan x$ graphically (pp. 6, 45, 47), and we have used rectangular axes on p. 37.

y is said to be an *explicit function* of x, when both y and x vary, but y and x are connected by an equation, such that when a value of x is known the corresponding value of y can be determined. This statement is written $y = f(x)$. Other letters (F, E, ϕ, ψ, &c.) are used, as well as f, when more than one function is involved in a problem.

If the actual arithmetic of evaluation can be performed, the values of $f(x)$ or y can be plotted by the ordinary method on squared paper. If, as x takes all possible values from $x = a$ to $x = b$, where a and b are any fixed quantities, the corresponding values of y are such that they can be represented by a line drawn without removing the pencil from the paper, and if to every point on the line drawn there corresponds a value of x, then the function is said to be *continuous* in x from a to b. [This should be regarded as a preliminary definition of continuity; a rigorous definition involves very difficult conceptions.]

The simplest functions of x are given in the equations

$$f(x) = y = x\,;\ y = mx\,;\ y = mx + k\,;\ y = ax^2 + bx + c\,;$$

where m, k, a, b, c are constants, that is, are unchanged while x and y vary.

Note. Throughout this section the conventions of pp. 37–8 are used, so that e.g. $MP = -PM$, and $LP = OP - OL$ for all possible positions of O, L, P in a line.

Direct Variation.

When $y = mx$, y is said to vary directly as x. This is written $y \propto x$, and m is called the constant of variation.

If (x_1, y_1) (x_2, y_2) $(x_3, y_3)\ldots$ are pairs of values, then

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3} = \ldots = m,$$

that is, the ratio of the constituents of a pair is constant.

If a line be drawn through the zero point of two axes at right angles to each other, on which x and y are measured to make an angle $\theta = \tan^{-1} m$ with OX, and any point P be taken on it, and PN, PM be drawn parallel to the axes OX, OY to meet the axes OY, OX, then from pp. 41–3

$$\frac{PM}{OM} = \frac{ON}{OM} = \tan\theta = m$$

for all positions of P (see Fig. 37).

[OM is called the *abscissa* (the part cut off), MP the *ordinate* of the point P. OM, MP together are called the *co-ordinates* of P with reference to the *axes of reference* OX, OY. OX is called the axis of x, OY the axis of y.]

Hence if (x_1, y_1) are the co-ordinates of any point P on the line,

$$\frac{y_1}{x_1} = m.$$

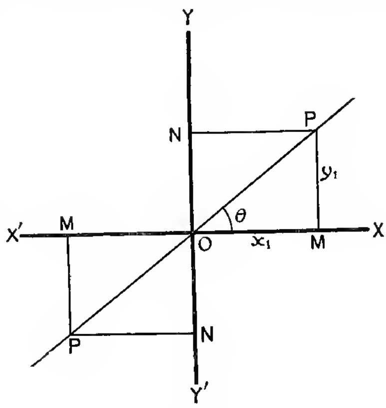

FIG. 37.

Conversely, any pair of values that satisfies the relation $y = mx$ can be represented by a point on the line.

The straight line OP, continued indefinitely in both directions, is then said to represent the equation $y = mx$, and $y = mx$ is said to be the equation of the line.

If (x_0, y_0) be a pair of standard or known values, then the value of y corresponding to any value of x is found from the equation $\dfrac{y}{x} = m = \dfrac{y_0}{x_0}$, $y = \dfrac{y_0}{x_0} \times x$.

A relation, such as $y - 4 = 3(x - 5)$ or $y = 3x - 11$, can be put in the form $y = mx + k$. Such a relation is direct variation, where the y and x are measured, not from zero, but as the excess, above fixed values; for example, if the charge for excess luggage is 3 farthings for every lb. above 100, together with a registration charge of sixpence, the equation $y = 24 + 3(x - 100)$ would give the charge (y farthings) for x lb. of luggage ($x > 100$).

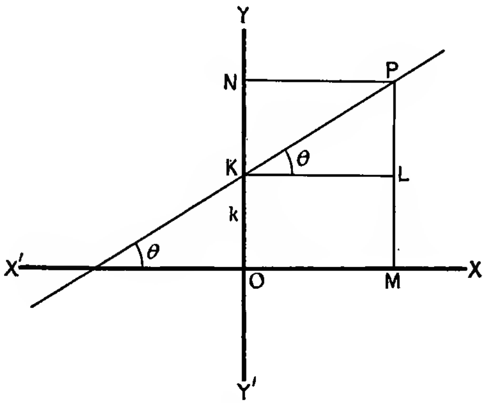

FIG. 38.

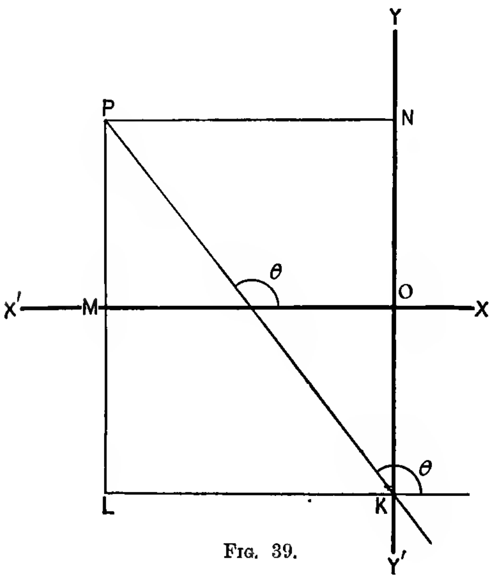

FIG. 39.

Mark off $OK = k$ on the axis of y (Fig. 38 or Fig. 39). Through K draw a line making an angle $\theta = \tan^{-1} m$ with the axis of x. Take any point P on this line, draw PM, PN as before, and draw KL parallel to OX to meet MP.

Then for all positions of P and of K (whether OK is positive or negative), using the convention that MP, &c., involve direction and magnitude as on p. 36,

$$m = \tan\theta = \frac{KN}{KL} = \frac{KO + ON}{KL} = \frac{ON - OK}{OM} = \frac{y-k}{x},$$

where (x, y) are the co-ordinates of P. $\therefore$ $y = mx + k$, and the co-ordinates of any point on the line drawn satisfy the relation. Conversely, any pair of values (x, y) which satisfy the relation is represented by a point on the line.

We have here shown implicitly that the equation $y = mx + k$ is the equation of a straight line. (See p. 133.)

The Quadratic Function.

The equation $y = ax^2 + bx + c$ is closely connected with the theory of quadratic equations, which we shall summarize here; it is also a form of the equation of the conic section called a parabola (see pp. 148, 153), and has other applications.

The graph of y depends of course on the magnitudes of a, b, c. For simplicity we shall take a to be positive. The student is recommended to work through the arguments also with a negative, finding the maximum value of y.

$$y = ax^2 + bx + c = a\left(x^2 + 2\cdot\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2\right) + c - a\left(\frac{b}{2a}\right)^2$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]. \quad . \quad . \quad . \quad . \quad \text{(i)}$$

When x is a very great quantity, positive or negative, y is very great.

The least value y can have is obtained when $\left(x + \frac{b}{2a}\right)^2 = 0$, i.e. when $x = -\frac{b}{2a}$, for this square cannot be negative. This minimum value of y is $\frac{4ac-b^2}{4a}$.

The minimum value of y is positive if $4ac > b^2$, zero if $4ac = b^2$, negative if $b^2 > 4ac$.

The curve representing the equation can be shown to be symmetrical about the vertical line through its lowest point (V), i.e. the point where y is least and $x = -\frac{b}{2a}$.

For cut off $OM = -\frac{b}{2a}$ (see Figs. 40–43). Then

$$MV = \frac{4ac-b^2}{4a}.$$

Let P be any point on the curve, whose abscissa is OL.

Then, from (i),

$$LP = y = a\left\{(OL-OM)^2 + \frac{4ac-b^2}{4a^2}\right\} = a\,.\left(ML^2 + \frac{4ac-b^2}{4a^2}\right).$$

Now take L' so that M is the middle point of LL' and $ML + ML' = 0$. Draw the ordinate $L'P' = LP$.

Let (x', y') be the co-ordinates of P', so that $y' = L'P'$, $x' = OL'$.

Then $(ML)^2 = (-ML)^2 = (ML')^2 = (OL'-OM)^2 = \left(x' + \frac{b}{2a}\right)^2$,

and $y' = LP = a\left\{\left(x' + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right\} = ax'^2 + bx' + c.$

$\therefore$ the co-ordinates of P' satisfy equation (i) and P' is on the curve.

Then by obvious geometry PP' is bisected (at N) at right angles by the vertical through M. Hence the points of the curve on lines parallel to OX are in pairs equally distant from VM; that is, the curve is symmetrical with regard to VM.

$y = 2x^2 - 4x + 1$

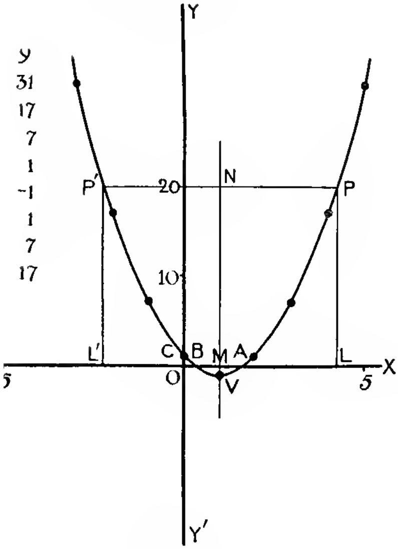

Fig. 40.

$y = 3x^2 - 12x - 2$

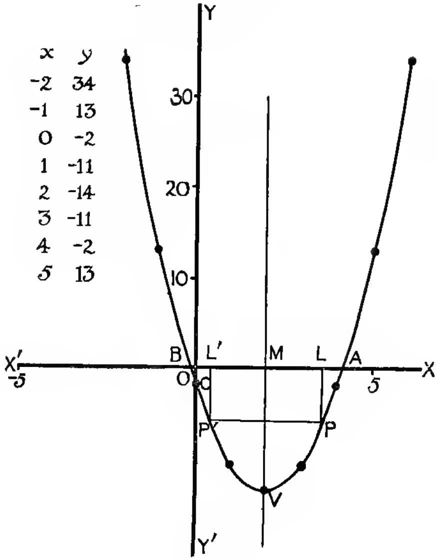

Fig. 41.

$y = 2x^2 + 13x + 18$

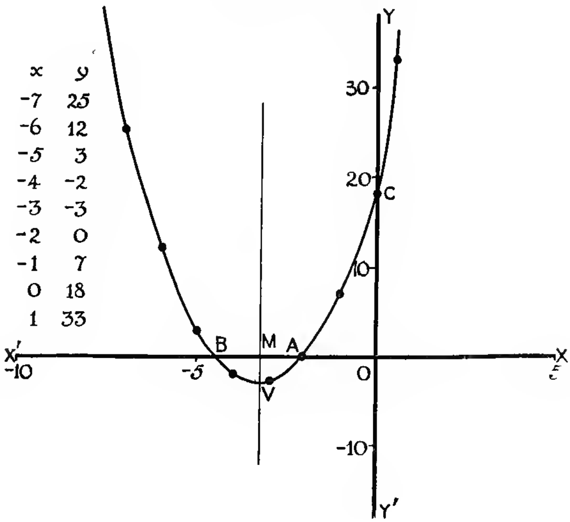

Fig. 42.

If $b^2 < 4ac$, V, the lowest point, is above OX, and there can be no intersection with OX (Fig. 43).

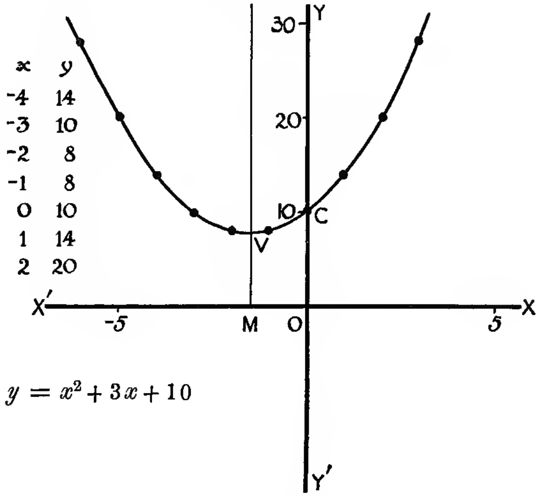

FIG. 43.

If $b^2 > 4ac$, V is below OX, and y is negative, when $x = -\frac{b}{2a}$ (Figs. 40–42). When x is great (+ or −) y is positive. If the numerical calculations are carried out for evaluating y, e.g. for the functions on the diagram, it becomes evident* that the function is continuous, and that the line representing the function can be drawn without taking the pencil from the paper. Hence, as the value of y changes from positive to negative and back again to positive as x varies, it must intersect the axis OX at two points A and B.

Let $OA = \alpha$, $OB = \beta$. Then α and β are values of x which make y zero, i.e. α and β are roots of the equation

$$ax^2 + bx + c = 0.$$

$\alpha + \beta = OA + OB = OM + MA + OM + BM = 2\,OM$, since MA and MB are, from the symmetry of the curve, equal and opposite.

* This is not a proof. As the examinee wrote, 'Proof is a hard word, but thank heaven I have made the proposition plausible.' That A and B exist, if $b^2 > 4ac$, is shown by solving the equation as on p. 79.

$$\therefore\ \alpha+\beta = 2 \times \left(-\frac{b}{2a}\right) = -\frac{b}{a}.$$

Also $\alpha\beta = (OM + MA)(OM + MB) = OM^2 - MA^2$, since $MB = -MA$.

Now $x = \alpha$, $y = 0$ satisfies the equation of the curve.

$$\therefore\ 0 = a\left\{(OA - OM)^2 - \frac{b^2-4ac}{4a^2}\right\}.$$

$$\therefore\ \frac{b^2-4ac}{4a^2} = MA^2.$$

$$\therefore\ \alpha\beta = \left(-\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2} = \frac{c}{a}.$$

[Of course these relations can be obtained by direct solution of the equation $cx^2+bx+a = 0$; for α and β are $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$; $\therefore\ \alpha+\beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$.]

When $x = 0$, $y = c$. In the figures OC is cut off equal to c.

By taking various values the diagrams show that when c as well as a is positive, α and β (OA and OB) are both positive (Fig. 40) or both negative (Fig. 41); when c is negative, α and β are of opposite signs (Fig. 42).

If $b^2 > 4ac$, $y = a\left\{x^2 + \dfrac{b}{a}x + \dfrac{c}{a}\right\} = a\,\{x^2-(\alpha+\beta)\,x+\alpha\beta\}$

$$= a\,(x-\alpha)\,(x-\beta).$$

Hence $x-\alpha$ and $x-\beta$ are factors of ax^2+bx+c, when these are real roots.

Also (taking $\alpha > \beta$),

when $x > \alpha$, y is positive, both factors being positive;

$x = \alpha$, $y = 0$;

$\alpha > x > \beta$, y is negative, the first factor being negative;

$x = \beta$, $y = 0$;

$x < \beta$, y is positive, both factors being negative.

Thus the change in sign of y can be traced and compared with the figures as x diminishes from a large positive to a large negative quantity (Figs. 40–42).

If $b^2 < 4ac$, $y = a\left\{\left(x + \dfrac{b}{2a}\right)^2 + \dfrac{4ac-b^2}{4a^2}\right\}$ and is always positive, as in Fig. 43.

Inverse Variation.

The following are simple functions of x, where x appears in the denominator of a fraction.

$$f(x) = y = \frac{a}{x};\ y = \frac{a+bx}{x};\ y = \frac{a+bx}{x+d};\ y = \frac{a+bx+cx^2}{x+d}.$$

When $y = \frac{a}{x}$, a being constant, y is said to vary inversely as x. This is written $y \propto \frac{1}{x}$.

If y and x are measured, not from zero, but as the excess above fixed values, we have such an equation as $y-4 = \frac{3}{x-5}$, i.e. $y = 4 + \frac{3}{x-5} = \frac{4x-17}{x-5}$.

Any such relation can be put in the form $y = \frac{a+bx}{x+d}$, which reduces to $y = \frac{a+bx}{x}$ when d is 0, and to $y = \frac{a}{x}$ when b also is 0. It will therefore be sufficient to trace the graph of the more complex function only. It will readily be seen that none of the essential properties of the graph depend on the numerical values or on the signs taken.

$$y = f(x) = \frac{4x-17}{x-5} = 4 + \frac{3}{x-5}.$$

x	y	x	y	
$-\infty$ *	4	1	3.25	
-10	3.80	2	3	
-8	3.77	3	2.5	
-6	3.73	4	1	
-4	3.67	$5-h$	$4-\frac{3}{h}$	where h is any small quantity.
-2	3.57	$5+h$	$4+\frac{3}{h}$	
0	3.4	6	7	
$4\frac{1}{4}$	0	7	5.5	
$4\frac{1}{2}$	-2	10	4.6	
$5\frac{1}{2}$	10	∞ *	4	

As x approaches 5 from either side, y becomes indefinitely great.

* The sense in which this is used is explained in Section V, p. 101.

Thus if $x = 5 - \frac{1}{10}$, $y = 4 - \dfrac{3}{\frac{1}{10}} = -26$,

$x = 5 - \frac{1}{100}$, $y = 4 - 300 = -296$,

$x = 5 + \frac{1}{10}$, $y = 4 + 30 = 34$,

$x = 5 + \frac{1}{100}$, $y = 4 + 300 = 304$.

Let C be the point (5, 4), Figure 44.

If P_1 be any point on the curve whose abscissa is in the neighbourhood of 5, and $P_1 M$, parallel to OX, meet OY in M and the line CR ($x = 5$) in R,

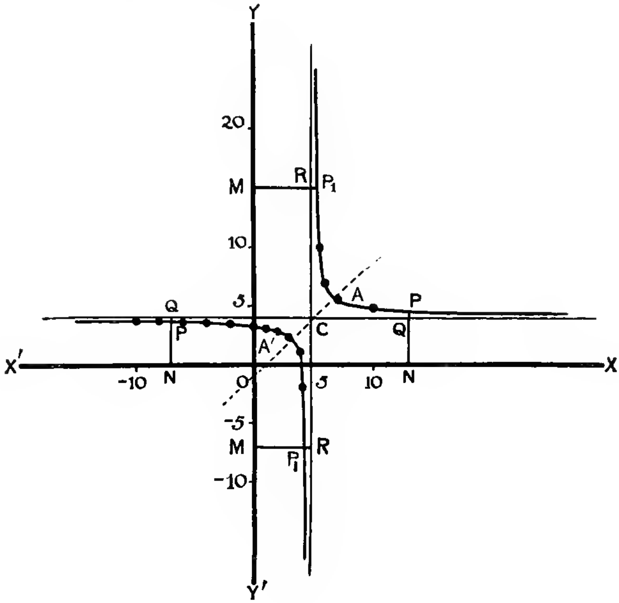

Fig. 44.

then $OM = y = 4 + \dfrac{3}{x-5} = 4 + \dfrac{3}{MP_1 - MR} = 4 + \dfrac{3}{RP_1}$,

where RP_1 may be positive or negative.

As OM is positively increased indefinitely, RP_1 becomes very small and remains positive. As OM becomes a great negative quantity, RP_1 becomes very small and negative. The curve therefore lies indefinitely near to either side of the line CR produced both ways.

Similarly, if P be a point whose ordinate is nearly 4, and PQN, parallel to OY, meet OX in N and the line $y = 4$ (CQ) in Q,

$$\text{then } QP = NP - NQ = y - 4 = \frac{3}{x-5} = \frac{3}{ON-5}.$$

As ON increases indefinitely, QP becomes indefinitely small.

Lines as CR, CQ which satisfy the conditions described are called *asymptotes* (lines which do not meet) to the curve.

The curve now drawn is a *rectangular hyperbola.*

It is left as an exercise to show that the two branches are equal in all respects.

As a guide to drawing, it may be stated that the line through C at 45° to OX ($A'CA$) divides the curve symmetrically (see pp. 147, 148).

In the ordinary case of inverse variation, $y = \frac{a}{x}$, the pairs of values are connected by the equations $x_1 y_1 = x_2 y_2 = \ldots = a$.

If $x_0 y_0$ be a pair of standard or known values, the value of y corresponding to any x is $y = \frac{x_0 y_0}{x}$.

Compound variation. The more complicated cases of variation, involving more than one variable, are best discussed by an example.

If V is the volume of a given amount of gas, P the pressure under which it is kept, and T its temperature on the scale of absolute temperatures, then the following physical laws are true:

$$V \propto T, \text{ if } P \text{ is unchanged,}$$

$$V \propto \frac{1}{P}, \text{ if } T \text{ is unchanged.}$$

Let P_0, T_0 be standard pressure and temperature. These are generally taken as $P_0 = 760$ (mm.), the mean atmospheric pressure, and $T_0 = +273$ (degrees), the temperature of freezing-point on the absolute scale.

Suppose P unchanged while the temperature changes to T_0, and let V_1 be the resulting volume. Then $\dfrac{V}{T} = \dfrac{V_1}{T_0}$.

Now suppose the temperature unchanged at T_0, while P changes to P_0, and let V_0 be the resulting volume.

Then $V_1 . P = V_0 P_0$.

Eliminate V_1 from these equations and we have $\dfrac{VP}{T} = \dfrac{V_0 P_0}{T_0}$.

V_0, being the volume at standard pressure and temperature, may be called the standard volume.

Hence under all changes of temperature and pressure, $\dfrac{VP}{T}$ remains unchanged.

$$\therefore \quad V = \frac{T}{P} \times \text{const.} \propto \frac{T}{P} \propto T \times \frac{1}{P}.$$

Similarly given that the volume (V) of a right circular cone varies as its altitude (h) when the radius (r) of its base is constant, and as the square of its base radius when its altitude is constant, it follows that

$$V = hr^2 \times \text{const.} \propto h \times r^2.$$

In the latter case V is said to vary conjointly with h and the square of r; in the former case V varies conjointly with T directly and P inversely.

Notice that in neither case is the 'constant of variation' used. In experiments the former is used as the equation

$$V_0 = V \times \frac{P}{P_0} \times \frac{T_0}{T}$$

Generally, if $x \propto y$ when z is constant, and $x \propto z$ when y is constant, then $x \propto y \times z$ when y and z both vary.

The graph of the function $\dfrac{a + bx + cx^2}{x + d}$ illustrates several important methods.

One numerical example will be sufficient.

Let $y = \dfrac{3x^2 - 12x + 35}{2x - 10} = \frac{3}{2}x + \frac{3}{2} + \dfrac{25}{x-5}$ by direct division.

x	y
0	-3.50
1	-3.25
2	-3.83
3	-6.50
4	-17.5
$4\frac{1}{2}$	-41.75
$5-h$	$9-\frac{3}{2}h-\frac{25}{h}$
$5+h$	$9+\frac{3}{2}h+\frac{25}{h}$
$5\frac{1}{2}$	59.7
6	35.5
7	24.5
8	21.83
9	21.25
10	21.5
12	23.1
14	25.3
16	27.8
⋮	
-1	-4.17
-2	-5.07
-3	-6.1
-4	-7.3
-5	-8.5
-6	-9.8
-8	-12.4
-10	-15.2
$+\infty$	$+\infty$
$-\infty$	$-\infty$

Since $3x^2-12x+35=0$ has no real roots, y is never zero.

Let P be any point on the curve. Let PM, parallel to OY, meet OX in M, and the line $y=\frac{3}{2}x+\frac{3}{2}$ in Q, so that $MQ=\frac{3}{2}OM+\frac{3}{2}$.

Then

$$MP=y=\tfrac{3}{2}x+\tfrac{3}{2}+\frac{25}{x-5}=MQ+\frac{25}{x-5}.$$

$$\therefore\ QP=\frac{25}{x-5}.$$

$x=5$ is clearly an asymptote.

As x increases indefinitely QP becomes small, and CQ $(y=\frac{3}{2}x+\frac{3}{2})$ is an asymptote, where C is $(5, 9)$ the intersection of $x=5$, and

$$y=\tfrac{3}{2}x+\tfrac{3}{2}.$$

The curve is then as drawn in Figure 45.

This curve is a hyperbola (see pp. 147, 151).

Whatever value of x is taken, there is one and only one value of y. Every vertical line meets the curve once.

But if a value of y is taken, say y_1, we have a quadratic, $3x^2-12x+35=y_1(2x-10)$ to determine x.

The roots of this are real,

if $(-6-y_1)^2-3(35+10y_1) \nless 0$,

if $y_1{}^2-18y_1-69 \nless 0$,

if $(y_1-\alpha)(y_1-\beta) \nless 0$, where $\alpha=9+5\sqrt{6}=21{\cdot}25$, and $\beta=9-5\sqrt{6}=-3{\cdot}25$ approx.

If $y_1>\alpha$ both factors are positive, and if $y_1<\beta$ both factors are negative; in both cases the roots are real and different. That is, every horizontal line above $21{\cdot}25$ or below $-3{\cdot}25$ meets the curve in two points, as LK, $L'K'$ in Figure 45.

If $y_1=\alpha$ or β the roots in x are equal, being $9{\cdot}1$ and ${\cdot}9$ (approx.) respectively, and $y_1=21{\cdot}25$, $y_1=-3{\cdot}25$ are horizontal tangents (touching at D, E).

If y_1 is between α and β, $(y_1-\alpha)$ is negative, $(y_1-\beta)$ positive, the product is positive, and there are no real roots in x.

Hence y_1 can have all values except those between $21{\cdot}25$ and $-3{\cdot}25$.

The result is expressed thus: the *range* of the function is from $-\infty$ to $-3{\cdot}25$ and from $21{\cdot}25$ to $+\infty$.

Similarly the range of the function $2x^2 - 4x + 1$ (Fig. 40, p. 77) is from -1 to $+\infty$.

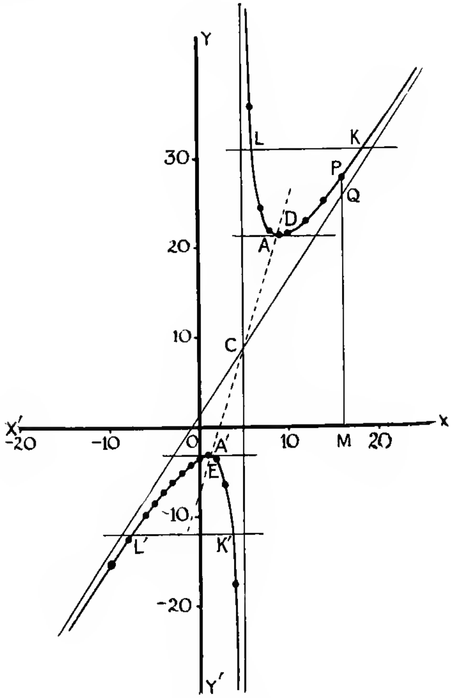

Fig. 45.

(The line AA' bisecting the angle between the asymptotes divides the curve symmetrically, see p. 147.)

Examples.

1. If $y = z^3$ and $z^2 = x^5$, express y as a function of x.
If $p = \cos(q + x)$ and $\cos x = q$, express p as a function of q only.

2. Show that $3x^2 - 7x + 53$ has a minimum value of $48\frac{11}{12}$.
" " $27 - 2p - p^2$ has a maximum value of 28.
" " $30y^2 + 27y - 154$ has a minimum value when $y = -\frac{9}{20}$.
Test your results graphically.

3. If y varies as x, and x varies as z, and $y = 3$ when $x = 10$, and $x = -2$ when $z = 6$, find z when $y = -1$, and show that z always equals $-10y$.

4. The horse-power required to drive a given ship varies directly as the cube of the speed. If 5,000 H.-P. are required at 29 knots, show that 850, roughly, will be required at 16 knots.

5. The intensity of illumination at a point due to a source of light varies inversely as the square of the distance of the light from that point. If a person can just see to read a book 4 ft. from a candle, show that he will require about 8 such candles together to see to read 11 ft. away from them.

The Rational Integral Function. Solution of Equations.

If $y = f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, where n is a positive integer and $a_n, a_{n-1}, \dots a_1, a_0$ are real commensurable quantities, $f(x)$ is said to be a *Rational Integral Function* of x of the n^{th} degree.

Remainder Theorem.

Divide $f(x)$ by $x - \alpha$ in the ordinary algebraic way, till the remainder, R, does not contain x. The quotient, Q, is a rational integral function of the $\overline{n-1}^{\text{th}}$ degree, whose first term is $a_n x^{n-1}$.

Then $f(x) = (x-\alpha) \,.\, Q + R$, whatever the value of x.

[Thus $2x^3 - 3x + 5 = (x-2) \,.\, (2x^2 + 4x + 5) + 15$. Here

$$Q = 2x^2 + 4x + 5. \ R = 15.]$$

The identity is still true if $x = \alpha$. In this case

$$R + (\alpha - \alpha) \,.\, Q' = f(\alpha) = a_n \alpha^n + a_{n-1} \alpha^{n-1} + \dots a_0,$$

where Q' is the value of Q when α is substituted for x.

$$\therefore \ R = f(\alpha).$$

$$[R = 2 \times 2^3 - 3 \times 2 + 5 = 15.]$$

Hence the remainder, when $f(x)$ is divided by $x - \alpha$, is obtained by writing α for x in $f(x)$.

Corollary. If α is a root of the equation $f(x) = 0$, then $R = f(\alpha) = 0$, and $x - \alpha$ is an exact factor of $f(x)$.

Roots of f (x) = 0.

To show that *there cannot be more than* n *different real roots of* $f(x) = 0$.

If possible let $\alpha_1, \alpha_2 \ldots \alpha_n, \alpha_{n+1}$ be roots, none equal to another.

Then $f(x) = (x-\alpha_1) \,.\, Q = (x-\alpha_1)\{(x-\alpha_2)\, Q_1 + R_1\}$, where Q_1 is quotient and R_1 the remainder if Q is divided by $x-\alpha_2$. The first term of Q_1 is $a_n x^{n-2}$.

Writing α_2 for x, $\quad 0 = f(\alpha_2) = (\alpha_2 - \alpha_1)\{0 + R_1\}$.

$\therefore\ R_1 = 0$, unless $\alpha_2 = \alpha_1$.

$\therefore\ f(x) = (x-\alpha_1)(x-\alpha_2) \,.\, Q_1$.

Continuing this process, we have

$$f(x) = a_n (x-\alpha_1)(x-\alpha_2)(x-\alpha_3) \ldots (x-\alpha_n).$$

This is true for all values of x, e.g. when $x = \alpha_{n+1}$,

$\therefore\ 0 = f(\alpha_{n+1}) = a_n (\alpha_{n+1}-\alpha_1)(\alpha_{n+1}-\alpha_2) \ldots (\alpha_{n+1}-\alpha_n)$.

But unless one of the factors is zero the product cannot be zero.

$\therefore\ a_n = 0$, or $\alpha_{n+1} = \alpha_1$ or $\alpha_2 \ldots$ or α_n.

Hence α_{n+1} equals one of the roots already used, unless $a_n = 0$.

If $a_n = 0$, the function is only of the $\overline{n-1}^{\text{th}}$ degree. Applying the same argument it would follow successively that $a_{n-1} = 0$, $a_{n-2} = 0 \ldots a_0 = 0$, and the function disappears.

Hence, unless the function is identically zero, there cannot be more than n roots of $f(x) = 0$.

It does not follow that there are as many as n or indeed any roots of the equation; this is considered in Section VIII, p. 238.

Conjugate Roots.

In the proof of the remainder theorem, it was not assumed that α was commensurable.

It can be shown that if $\beta + \sqrt{\gamma}$ is a root of $f(x) = 0$, then $\beta - \sqrt{\gamma}$ is also a root, where γ is positive and not a perfect square, and β and γ commensurable.

Divide $f(x)$ by

$$\{x - (\beta + \sqrt{\gamma})\}\{x - (\beta - \sqrt{\gamma})\} = x^2 - 2\beta x + \beta^2 - \gamma.$$

Let Q be the quotient.

The remainder may be written $R_1 x + R$, where R_1 and R do not contain x, and are commensurable since all terms in the divisor are commensurable.

Then $$f(x) = Q \cdot (x^2 - 2\beta x + \beta^2 - \gamma) + R_1 x + R.$$

Substitute $\beta + \sqrt{\gamma}$ for x and let Q' be the value taken by Q;

$$0 = f(\beta + \sqrt{\gamma}) = Q' \times 0 + R_1(\beta + \sqrt{\gamma}) + R.$$

$\therefore$ $R_1 \sqrt{\gamma} = -(R + R_1 \beta)$, commensurable equal to incommensurable, which is only possible when each is 0.

$\therefore$ $R_1 \sqrt{\gamma} = 0$, $R_1 = 0$, and $0 = R + R_1 \cdot \beta = R$.

$\therefore$ $f(x) = Q \cdot \{x - (\beta + \sqrt{\gamma})\}\,\{x - (\beta - \sqrt{\gamma})\}$ for all values of x.

Write $\beta - \sqrt{\gamma}$ for x,

then $$f(\beta - \sqrt{\gamma}) = Q'' \cdot (-2\sqrt{\gamma}) \cdot 0 = 0;$$

$$\therefore\ \beta - \sqrt{\gamma} \text{ is a root.}$$

$\beta + \sqrt{\gamma}$ and $\beta - \sqrt{\gamma}$ are said to be conjugate.

[This result is already familiar in quadratics; e.g. the roots of $x^2 - 3x + 1 = 0$ are $\frac{3}{2} \pm \sqrt{\frac{5}{4}}$.]

If we knew that such an identity as $(x+a)^3 = x^3 + 6x^2 + bx + c$ was true independently of the value of x, we should probably not hesitate to equate the coefficients of equal powers of x on the two sides and say $3a = 6$, $3a^2 = b$, $a^3 = c$, and $a = 2$, $b = 12$, $c = 8$. The following is the justification for the process.

THEOREM. *If* $$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$$
and $$b_n x^n + b_{n-1} x^{n-1} + \ldots b_0,$$
two rational integral functions, are equal for more than n *values of* x, *then the coefficients are equal term by term and the functions are equal for all values of* x.

For let $\alpha_1, \alpha_2, \ldots \alpha_{n+1}$ be the values of x for which the functions are equal. Then these quantities are $n+1$ roots of the equation

$$(a_n - b_n)\,x^n + (a_{n-1} - b_{n-1})\,x^{n-1} + \ldots + (a_1 - b_1)\,x + (a_0 - b_0) = 0.$$

As on p. 87, two of the α's are equal, contrary to the hypothesis, or $a_n - b_n = 0$, and successively

$$a_{n-1} - b_{n-1} = 0,\ a_{n-2} - b_{n-2} = 0 \ldots .$$

$\therefore$ $a_n = b_n$, $a_{n-1} = b_{n-1}$, $a_{n-2} = b_{n-2} \ldots$, $a_1 = b_1$ and $a_0 = b_0$.

Hence the functions are equal to each other for all values of x. Of course n is assumed to be finite in this theorem.

This theorem is applied to establish certain *relations between the coefficients and roots of an equation* of the nth degree.

Let $\alpha_1, \alpha_2 \ldots \alpha_n$ be n different roots (if possible) of $f(x) = 0$.

Then by p. 87,

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x + a_0 = f(x)$$
$$= a_n (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$$
$$= \text{by direct multiplication}$$
$$a_n \{x^n - x^{n-1}(\alpha_1 + \alpha_2 +) + x^{n-2}(\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \ldots + \alpha_1 \alpha_n) - \ldots\}$$
$$= a_n \{x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots + (-1)^n p_n\},$$

where p_1 is the sum of the n roots, p_2 is the sum of the ${}_nC_2$ products of roots 2 at a time, p_3 the sum of the ${}_nC_3$ products 3 at a time, ... and p_n is the product of all the roots. E.g. in the equation of the 3rd degree $p_1 = \alpha + \beta + \gamma$, $p_2 = \alpha\beta + \beta\gamma + \gamma\alpha$, $p_3 = \alpha\beta\gamma$ where α, β, γ are the roots.

This equation is true for all, that is, for more than n, values of x. Hence the coefficients on the two sides are equal, each to each.

$$\therefore\ p_1 = -\frac{a_{n-1}}{a_n},\ p_2 = \frac{a_{n-2}}{a_n},\ p_3 = -\frac{a_{n-3}}{a_n} \ldots p_n = (-1)^n \frac{a_0}{a_n}.$$

The theorem is easily interpreted if two or more of the roots are equal.

E.g. in the quadratic $ax^2 + bx + c = 0$, $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$, where α, β are the roots (see p. 79).

If $\alpha = \beta$, $2\alpha = -\frac{b}{a}$, $\alpha^2 = \frac{c}{a}$, and $b^2 = 4ac$.

NOTE. These relations are not of use in solving equations in general.

In some cases where we have further information as to the roots, we can reduce the degree of the equation and then solve it.

E.g. given that the roots of $8x^3 - 12x^2 - 2x + 3 = 0$ are in arithmetic progression;

Let $\alpha - d$, α, $\alpha + d$ be the roots—

$$\tfrac{12}{8} = \text{sum of roots} = 3\alpha;\quad \therefore\ \alpha = \tfrac{1}{2}$$
$$-\tfrac{3}{8} = \text{product of roots} = \alpha(\alpha^2 - d^2) = \tfrac{1}{2}(\tfrac{1}{4} - d^2);$$
$$\therefore\ 4d^2 = 1 + 3,\ d = \pm 1, \text{ and the roots are } -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}.$$

Allied Equations.

Multiplication of Roots.

If α is a root of $f(x) = 0$, then $r\alpha$ is a root of

$$F(x) = a_n x^n + r a_{n-1} x^{n-1} + r^2 a_{n-2} x^{n-2} + \ldots + r^{n-1} a_1 x + r^n a_0 = 0$$

where r is any quantity. For

$$F(r\alpha) = a_n r^n . \alpha^n + r a_{n-1} r^{n-1} \alpha^{n-1} + \ldots + r^{n-1} a_1 r\alpha + r^n a_0,$$
$$= r^n f(\alpha) = 0.$$

In particular, take $r = -1$.

Then if α is a root of $f(x) = 0$, $-\alpha$ is a root of

$$a_n x^n - a_{n-1} x^{n-1} + \ldots + (-1)^n a_0 = 0.$$

Example. 2 and 3 roots of $x^2 - 5x + 6 = 0$; 20 and 30 are roots of $x^2 - 50x + 600 = 0$; $\cdot 2$ and $\cdot 3$ are roots of $x^2 - \cdot 5x + \cdot 06 = 0$; and -2, -3 are roots of $x^2 + 5x + 6 = 0$.

Diminution of Roots.

Let $x' = x - d$, where d is any quantity.

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$

$$= A_n x'^n + A_{n-1} x'^{n-1} + \ldots + A_0 = F(x)$$

for all values of x.

A_n, A_{n-1}, ... A_0 can be found in terms of d by a direct process, which is very laborious except in the simplest cases.

E. g. if $d = 2$, $3x^3 + 2x^2 + 4x + 1$

$$= 3(x-2)^3 + 18x^2 - 36x + 24 + 2x^2 + 4x + 1$$
$$= 3(x-2)^3 + 20(x-2)^2 + 48x - 55$$
$$= 3(x-2)^3 + 20(x-2)^2 + 48(x-2) + 41.$$

They can be found more rapidly as follows:

$F(x)$ is merely a re-arrangement of $f(x)$; hence if $f(x)$ and $F(x)$ are both divided by $x - d$ the quotients must be the same and the remainders the same.

Hence the quotient when $f(x)$ is divided by $x - d$ is that when $F(x)$ is divided by x', viz.

$$A_n x'^{n-1} + A_{n-1} x'^{n-2} + \ldots + A_2 x' + A_1 = Q_1,$$

say, and the remainder is A_0. [Hence $A_0 = f(d)$, p. 86.]

Similarly, the quotient when Q_1 is divided by x' is

$$A_n x'^{n-2} + A_{n-1} x'^{n-3} + \ldots + A_2 = Q_2,$$

and the remainder is A_1, and so on.

Therefore $A_0, A_1, A_2, \ldots A_{n-1}$ are the remainders when $f(x)$ or $F(x)$ is divided by $x-d$ $\overline{n-1}$ times in succession, and $A_n = a_n$ since these are the coefficients of x^n.

Finally, if α is a root of $f(x) = 0$, then

$0 = f(\alpha) = F(\alpha) = A_n(\alpha - d)^n + A_{n-1}(\alpha - d)^{n-1} + \ldots + A_0,$

$\therefore$ $\alpha - d$ is a root of $A_n x^n + A_{n-1} x^{n-1} + \ldots + A_0 = 0$.

EXAMPLE. $d = 2$,

$$\begin{array}{l} x-2)\,3x^3+2x^2+4x+1\,(\,3x^2+8x+20 = Q_1 \\ \quad\ \underline{3x^3-6x^2} \\ \qquad\quad 8x^2+\ \ 4x \\ \qquad\quad \underline{8x^2-16x} \\ \qquad\qquad\quad 20x+\ \ 1 \\ \qquad\qquad\quad \underline{20x-40} \\ \qquad\qquad\quad A_0 = 41 \end{array}$$

$$\begin{array}{l} x-2)\,3x^2+8x+20\,(\,3x+14 = Q_2 \\ \quad\ \underline{3x^2-6x} \\ \qquad\quad 14x+20 \\ \qquad\quad \underline{14x-28} \\ \qquad\quad A_1 = 48 \end{array}$$

$$\begin{array}{l} x-2)\,3x+14\,(\,3 = A_3 \\ \quad\ \underline{3x-\ \ 6} \\ \quad\ A_2 = 20 \end{array}$$

$\therefore$ $3x^3 + 2x^2 + 4x + 1 = 3(x-2)^3 + 20(x-2)^2 + 48(x-2) + 41$, as found above.

The process of finding $A_0, A_1 \ldots A_n$ is greatly simplified, as follows, by Horner's method of division.

The numerical example just given may be written

$$\begin{array}{l} x-2\,)\,\overline{3x^3+2x^2}+4x+1\,(\,3x^2+8x+20 \\ \qquad\qquad\quad \underline{8x^2+4x} \\ \qquad\qquad\qquad\quad \underline{20x+1} \\ \qquad\qquad\qquad\qquad\quad 41 \end{array}$$

8 is obtained as $2 + 3 \times 2$ and then appears in the quotient.

$20 = 4 + \ \ 8 \times 2$

$41 = 1 + 20 \times 2.$

Omitting the x's we have

$$\begin{array}{llll} 3 & 2 & 4 & 1 \\ & 2\times 3 = 6 & 2\times 8 = 16 & 2\times 20 = 40 \\ \hline & 8 & 20 & 41 \end{array} \quad \text{or simply} \quad \begin{array}{cccc} 3 & 2 & 4 & 1 \\ & 6 & 16 & 40 \\ \hline & 8 & 20 & |\,41 \end{array}$$

Remainder 41 ; quotient $3x^2+8x+20 = Q_1$.

Similarly, to divide Q_1 by $x-2$,

$$\begin{array}{ccc} 3 & 8 & 20 \\ & 6 & 28 \\ \hline & 14 & |\,48 \end{array}$$

Remainder 48 ; quotient $3x+14 = Q_2$, and to divide Q_2 by $x-2$

$$\begin{array}{cc} 3 & 14 \\ & 6 \\ \hline & |\,20 \end{array}$$

Remainder 20; quotient 3.

Now put these divisions together

$$\begin{array}{cccc} 3 & 2 & 4 & 1 \\ & 6 & 16 & 40 \\ \hline & 8 & 20 & |\,41 \\ & 6 & 28 & \\ \hline & 14 & |\,48 & \\ & 6 & & \\ \hline & 20 & & \end{array}$$

We have $A_0 = 41$, $A_1 = 48$, $A_2 = 20$, $A_3 = 3$, and the roots of $3x^3+20x^2+48x+41 = 0$ are each 2 less than those of $3x^3+2x^2+4x+1$.

This abbreviated process becomes very easy with experience.

EXAMPLE. Reduce the roots of $x^4-3x^2+7x-1 = 0$ by 6

$$\begin{array}{ccccc} 1 & 0 & -3 & 7 & -1 \\ & 6 & 36 & 198 & 1230 \\ \hline & 6 & 33 & 205 & |\,1229 \\ & 6 & 72 & 630 & \\ \hline & 12 & 105 & |\,835 & \\ & 6 & 108 & & \\ \hline & 18 & |\,213 & & \\ & 6 & & & \\ \hline & 24 & & & \end{array}$$

Result $x^4+24x^3+213x^2+835x+1229 = 0$.

Numerical Solution of the General Equation in one unknown.

We have now all the materials for the approximate numerical solution of an equation $f(x) = 0$, where $f(x)$ is a rational integral function.

Consider $\quad f(x) = 3x^3 - 5x^2 - 11x - 10 = 0.$

$f(0) = -10$	$f(-1) = -7$
$f(1) = -23$	$f(-2) = -32$
$f(2) = -28$	
$f(3) = -7$	
$f(4) = +58$	

The function changes its sign between $x = 3$ and $x = 4$; the graph crosses the axis of x and there is a root between these values. It does not appear that there is any other root. (See graph, Fig. 46.)

Write a new equation with roots 3 less than those of $f(x) = 0$.

$$\begin{array}{rrrr}
3 & -5 & -11 & -10 \\
 & 9 & 12 & 3 \\ \hline
 & 4 & 1 & -7 \\
 & 9 & 39 & \\ \hline
 & 13 & 40 & \\
 & 9 & & \\ \hline
 & 22 & &
\end{array}$$

$$f(x) = 3x^3 - 5x^2 - 11x - 10$$
$$= 3x'^3 + 22x'^2 + 40x' - 7 = 0 = f_1(x'),$$

say, where $x' = x - 3$.

If P is any point on the curve and PMN is drawn to meet OY at N, and a new vertical axis O_1Y_1, through $x = 3$, at M then $PM = PN - MN = x - 3 = x'$.

The new equation is then referred to the axes O_1Y_1, O_1X.

$f_1(x')$ has a root in x' between 0 and 1.

Multiply the roots tenfold;

then $\quad f_2(x) = 3x^3 + 220x^2 + 4000x - 7000 = 0$

has a root between 0 and 10.

Figure 46*a* suggests that $f_2(1)$ is negative and $f_2(2)$ is positive, which is also at once evident by substitution. There is, therefore, a root of $f_2(x) = 0$ between 1 and 2, of $f_1(x) = 0$ between ·1 and ·2, and of $f(x) = 0$ between 3·1 and 3·2.

Now reduce the roots of $f_2(x)$ by 1, and again multiply by 10, obtaining an equation $f_3(x) = 0$. (See next page for the process.)

3	220	4000	−7000
	3	223	4223
	223	4223	−2767000
	3	226	
	226	444900	
	3		
	2290		

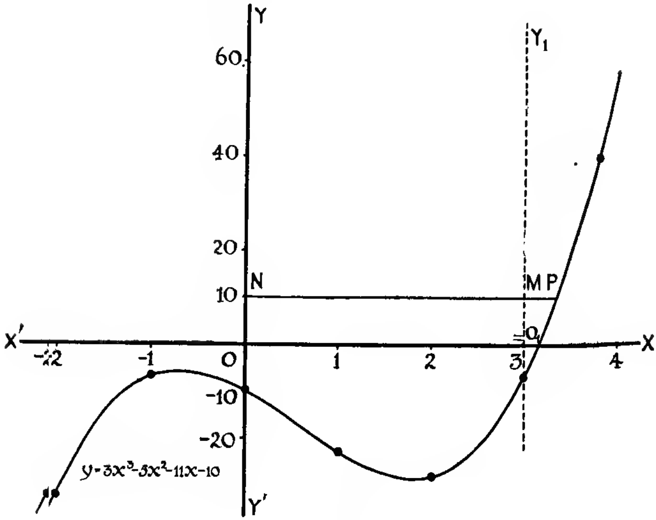

FIG. 46.

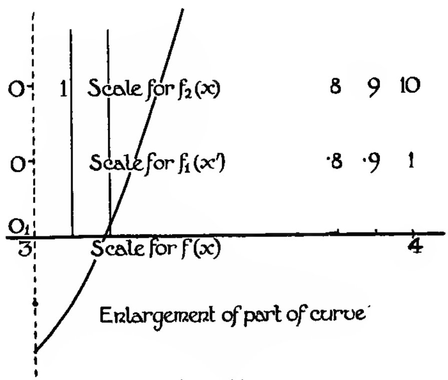

FIG. 46 *a*.

$$f_3(x) = 3x^3 + 2{,}290x^2 + 444{,}900x - 2{,}767{,}000 = 0.$$

$f_3(7)$ is positive, $f_3(6)$ is negative. The root of $f_3(x) = 0$ is between 6 and 7, of $f_2(x) = 0$ between 1·6 and 1·7, and of $f(x) = 0$ between 3·16 and 3·17.

This process can evidently be continued indefinitely; as shown in the next example the actual work is not very arduous when the method is mastered.

$f(0) = -\text{ve}$
$f(1) = -31$
$f(2) = +114$
......
$f(-1) = -\text{ve}$
$f(-7) = -\text{ve}$
$f(-8) = +\text{ve}$

To find one positive and one negative root of

$$x^4 + 9x^3 + 16x^2 + 23x - 80 = 0.$$

Roots between 1 and 2, and between −7 and −8.

Positive root

	1	9	16	23	−80 (1·3036
Reduce by 1		1	10	26	49
		10	26	49	−310000 (a')
		1	11	37	305691
		11	37	86000	−4309 (b')
		1	12	15897	
		12	4900	101897	
		1	399	17121	
(a)	1	130	5299	119018	
Reduce by 3		3	408		
		133	5707		
		3	417		
		136	6124		
		3			
		139			
		3			
(b)	1	142			
Multiply roots by 10					
(c)	·0001	·142	61·24	11901·8	−4309
(d)	0	0	61	11902	−4309(0
Multiply roots by 10					
(e)			·61	1190	−4309(36 by direct division.

If α is required root, the root of (a) (a') is $10(\alpha-1)=\alpha'$. Root of (b) (b') is $\alpha'-3$; if the multiplier 4 were taken the sign of the last term would be changed and the roots would have been passed.

The root of (c) is 10 times that of (b).

(d) is an approximation for c.

$+1$ changes the sign. Root is between 0 and 1.

The root of (e) is 10 times that of (d).

(d) is a quadratic equation. The next process reduces it to a simple equation and a final approximate number 6 can be found by division.

The root is then 1·3036 to the nearest digit in the fourth decimal place.

To find the *negative root*, multiply the roots by -1 (p. 90), i.e. write $-x$ for x. The positive root of

$$x^4-9x^3+16x^2-23x-80=0$$

is then the negative root of the given equation.

1	−9	16	−23	− 80	(7·4614
	7	−14	14	− 63	
	−2	2	−9	−1430000	
	7	35	259	1206016	
	5	37	250000	−223984	
	7	84	51504	218886	
	12	12100	301504	−5098	
	7	776	54672	3734	
	190	12876	356176	−1364	
	4	792	864		
	194	13668	36481		
	4	808	864		
	198	14476	37345		
	4				
	202				
	4				
·0001	206				

The negative root is then $-7·4614$ approx.

As a check, form the quadratic with roots 1·30 and $-7·46$, viz. $x^2+6·16\,x-9·70 = 0$.

This should be a nearly exact factor of

$$x^4+9\,x^3+16x^2+23x-80.$$

In fact this expression equals very nearly

$$(x^2+6·16x-9·70)\times(x^2+2·84\,x+8·25).$$

The other quadratic factor has no real factors. Hence there are no other real roots.

Many difficulties of handling arise in this method; they can generally be overcome by continually checking the work with the help of a careful graph. The processes of abbreviation should be studied by writing out all the work in full; it will then become obvious that abbreviation is possible and expedient. There is no simple way of determining how many real roots an equation has, but in ordinary cases the graph will give the necessary hints.

EXAMPLES.

1. Given that one root of $2x^3+17x^2-59x-120 = 0$ is $\frac{1}{2}(-7-\sqrt{209})$, show that another root is $-\frac{3}{2}$ and verify this by the Remainder Theorem.

2. Increase the roots of the quadratic equation $x^2+8x-5 = 0$ by 4 and thus solve it.

3. Solve the equation $x^3-5x^2+2x+8 = 0$, given that one root is double another.

4. Show that the equation $x^3-2x^2+7x-15 = 0$ has a root about $x = 2$. Prove that, more accurately, it is 2·089, and verify by a graph that it is the only real root.

5. Show that a root of the equation $x^4-4x^3+6x^2+8x-4 = 0$ is ·40631. Show that there is another real root.

6. Find an equation of the fourth degree two of whose roots are $3+\sqrt{7}$ and $5-\sqrt{2}$. Then solve the equation for the largest root by Horner's method.

7. Solve $x^6-84 = 0$.

8. Solve $2x^3-2x^2-27x+48 = 0$, and verify the roots obtained by the relations on p. 89.

SECTION V

LIMITS. SERIES

CONSIDER the progression $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}$.

The sum, S_n, $=\dfrac{\frac{1}{2}-\frac{1}{2^{n+1}}}{1-\frac{1}{2}}=1-\frac{1}{2^n}$. (p. 14.)

Whatever finite value n has, this sum is always less than 1. As n increases, the difference between 1 and S_n, viz. $\frac{1}{2^n}$, becomes smaller. If, as on pp. 25–6, we choose any small finite quantity, ϵ, we can make this difference less than ϵ,* by taking n so that $\frac{1}{2^n}<\epsilon$.

Thus $\frac{1}{2^n}<\epsilon$, if $n\log 2+\log\epsilon>0$, if $n>\dfrac{\log\frac{1}{\epsilon}}{\log 2}$. E.g. if ϵ were $\frac{1}{10^{20}}$ and $n>\frac{20}{\log_{10}2}$, $=664{\cdot}3\ldots$, the condition is satisfied, and S_{664} is between 1 and $1-\frac{1}{10^{20}}$.

The essential thing to notice is that however small ϵ is chosen to be, a *finite* value of n *can* be found (by an algebraic process in this case) so that S_n differs from 1 by less than ϵ.

In the curve $y=\dfrac{4x-17}{x-5}=4+\dfrac{3}{x-5}$ (p. 80), it was shown that as x increases y tends towards 4. The difference between y and 4 is less than ϵ, if $\frac{3}{x-5}<\epsilon$, that is, if $x>\frac{3}{\epsilon}+5$. Whatever

* Throughout this section ϵ means *as small a quantity as can be definitely assigned*, by any mental or numerical process.

(positive) value is assigned to ϵ, x can be chosen so that $y-4 < \epsilon$.

In considering $n \sin \frac{180^\circ}{n}$ on p. 65, we showed that, however great n was, this quantity was between 3·1405 and 3·1441.

The difference between the perimeters of two regular n-sided polygons, circumscribing and inscribed in a circle of unit diameter, is

$$n\left(\tan \frac{180^\circ}{n} - \sin \frac{180^\circ}{n}\right) = d \text{ (say)}.$$

Let θ be the radian measure of $\frac{180^\circ}{n}$ degrees, and let 1 radian $= k^\circ$, so that $\theta = \frac{180}{nk}$. Then

$$\frac{dk}{180} = \frac{1}{\frac{180}{nk}}\left(\tan \frac{180^\circ}{n} - \sin \frac{180^\circ}{n}\right) = \frac{1}{\theta}(\tan \theta - \sin \theta) < \frac{1}{\frac{2}{\theta^2} - 1}$$

$$d < \frac{180}{k\left(\frac{2}{\theta^2} - 1\right)}. \qquad \text{(p. 70)}$$

By increasing n we can make θ as small as we please, the denominator of the fraction last written as great and the fraction as small as we please. Hence we can choose a value for n such that d is less than any quantity (ϵ) assigned.

Hence columns [A] and [B] in the table of polygons on p. 65 can be continued till the difference between the perimeters of the external and internal polygons is less than ϵ. There is then a quantity between $n \sin \frac{180^\circ}{n}$ and $n \tan \frac{180^\circ}{n}$, such that the difference from either can be made as small as we please by increasing n. This quantity we defined to be π.

If successive values are given to x in the function $\frac{x^2 - a^2}{x - a} = y$, definite values can be found for y except when $x = a$, when numerator and denominator $= 0$. If x differs slightly from a

and $= a+h$, then $y = \dfrac{(a+h)^2-a^2}{a+h-a} = 2a+h$. As h diminishes, y approaches $2a$, and $y \sim 2a$* can be made $< \epsilon$ by taking $h < \epsilon$.

[E.g. $y = \dfrac{x^2-4}{x-2}$; $x = 2{\cdot}1$, $y = 4{\cdot}1$; $x = 2{\cdot}01$, $y = 4{\cdot}01$;

$x = 1{\cdot}99$, $y = 3{\cdot}99$.]

$$\frac{a+bx+cx^2}{A+Bx+Cx^2} - \frac{a}{A} = \frac{x(bA-aB)+x^2(cA-aC)}{A(A+Bx+Cx^2)}$$

$$= \frac{bA-aB+x(cA-aC)}{\dfrac{A^2}{x}+AB+ACx},$$

so long as x is not zero.

As x approaches zero, this fraction can be made $< \epsilon$ by increasing the denominator which contains the term $\dfrac{A^2}{x}$, unless $A = 0$.

Let $x = \dfrac{1}{u}$. Then $\dfrac{a+bx+cx^2}{A+Bx+Cx^2} = \dfrac{au^2+bu+c}{Au^2+Bu+C}$, unless $u = 0$.

This fraction can be made to differ from $\dfrac{c}{C}$ by less than ϵ, unless $C = 0$, by diminishing u, and therefore increasing x sufficiently.

The results now obtained are written

$$\operatorname*{Lt}_{n\to\infty}\left(\frac{1}{2}+\frac{1}{2^2}+\dots\frac{1}{2^n}\right) = 1, \quad \operatorname*{Lt}_{x\to\infty}\frac{4x-17}{x-5} = 4,$$

$$\operatorname*{Lt}_{n\to\infty} n\sin\frac{180^\circ}{n} = \operatorname*{Lt}_{n\to\infty} n\tan\frac{180^\circ}{n} = \pi = 3{\cdot}14159\ \dots,$$

$$\operatorname*{Lt}_{x\to a}\frac{x^2-a^2}{x-a} = 2a, \quad \operatorname{Lt}\frac{a+bx+cx^2}{A+Bx+Cx^2} = \frac{a}{A}\,(x\to 0) = \frac{c}{C}\,(x\to\infty).$$

The first is read 'the limit of the function as n tends to infinity is 1', and similarly for the others.

* $\sim$ is the usual abbreviation for 'the difference between' regarded as positive.

Notice that in these processes we have dealt entirely with intelligible finite algebraic quantities, and that no hazy ideas as to the nature of infinity are involved. 'Tends to infinity' is merely the conventional way of writing 'becomes greater than any finite quantity, however large, we like to choose'; ∞ is the abbreviation for the phrase 'as great a quantity as we can assign'.

In each case the variable (n or x) has been supposed to take successive values along some definite course, to increase or to diminish gradually, or to be equal to successive integers, in the direction of some assigned value, and that the final measurement of the function is taken before this value is reached, and it is shown that this measurement can be so taken that the function differs from a definite quantity by less than any numerical quantity, however small, that we can assign.

The formal definition of a limit is as follows:

If, when x approaches a value x_1, $f(x)$ approaches a value l, and if the difference between $f(x)$ and l is less than any small quantity we can assign (ϵ) for all values of x between $x_1 \pm h$, where h can be determined in terms of x_1 and ϵ, then l is called the limit of $f(x)$ when $x = x_1$; and if, when x is increased indefinitely, a quantity k can be found so that, when $x > k$, $f(x) \backsim l$ is less than ϵ, then l is called the limit of $f(x)$ when x is infinite.

EXAMPLE. Word the definition for the case when x is negative and increases indefinitely.

Some Important Limits.

$\underset{n \to \infty}{\mathrm{L}^t}\, r^n = 0$ if $r = 1 - d$, where d is between 0 and 1 and independent of n,* and n is integral.

Let $b = \dfrac{d}{1-d}$, and $\therefore\ 1 + b = \dfrac{1}{1-d}$. b is independent of n and positive.

* This is generally put less definitely as $0 < r < 1$; but it is essential that d should not tend to zero; e.g. $\mathrm{L}^t \left(1 \mp \frac{1}{n}\right)^n$ is not 0 (see p. 123).

$$r^n = (1-d)^n = \frac{1}{(1+b)^n} = \frac{1}{1+nb+\dots} \text{ (p. 23)} < \frac{1}{nb}.$$

However small the value of ϵ, n* can be taken $> \frac{1}{b\epsilon}$, and $\therefore \frac{1}{nb} < \epsilon$.

Hence by increasing n, $(r^n \backsim 0)$ can be made $< \epsilon$;

$$\therefore \mathop{\mathrm{Lt}}_{n\to\infty} r^n = 0.$$

$\mathop{\mathrm{Lt}}_{n\to\infty} \mathbf{Ar^n} = 0$, if $\mathbf{r = 1-d}$ as before, and A is constant.

For let $\epsilon = A\epsilon'$. However small ϵ, a quantity $\epsilon' = \frac{\epsilon}{A}$ can be assigned; then, as in the last case, r^n can be made less than ϵ' by increasing n, and $\therefore$ $Ar^n < \epsilon$.

[E.g. if A is 400, and ϵ is taken as $\frac{1}{10^{30}}$, then n must be taken so that $r^n < \frac{1}{400 \times 10^{30}}$.]

If $\mathop{\mathrm{Lt}} \mathbf{f(x)} = l$, then $\mathop{\mathrm{Lt}} \mathbf{Af(x)} = \mathbf{A}l$, where A is a constant.

For as in last article, $f(x) \backsim d$ can be made $< \frac{\epsilon}{A}$, and $\therefore$ $Af(x) \backsim Al$ can be made $< \epsilon$.

If $\quad \mathbf{S_n = a + ar + \dots + ar^{n-1}} = \frac{\mathbf{a\,(1-r^n)}}{\mathbf{1-r}}$ (p. 14), where $\mathbf{r = 1-d}$, $\mathbf{0 < d < 1}$ and d is constant (independent of n), then $\mathop{\mathrm{Lt}}_{n\to\infty} \mathbf{S_n} = \frac{\mathbf{a}}{\mathbf{1-r}}$.

For $S_n = \frac{a}{1-r} - \frac{a}{1-r} \cdot r^n$.

$\frac{a}{1-r} = \frac{a}{d} = A$, a constant quantity.

* A smaller value of n would serve; see note on p. 106.

$\therefore \dfrac{a}{1-r} - S_n = Ar^n < \epsilon$, if n is sufficiently increased.

[This is generally, but vaguely and erroneously, deduced in the elementary algebra of finite quantities; where $\dfrac{a}{1-r}$ is said to be the 'sum to infinity' (a vague and undefined expression) of a geometric progression.]

$\underset{n\to\infty}{\text{L}t}\,\dfrac{\mathbf{A}}{\mathbf{n}} = 0$ **whatever constant quantity A is.** This needs no proof.

$\underset{n\to\infty}{\text{L}t}\,\dfrac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} = 0$ **for all positive constant values of x.** Let x be an integer t, or between the integers t and $t+1$.

$$\frac{x^n}{n!} = \frac{x}{1}\cdot\frac{x}{2}\cdot\frac{x}{3}\cdots\frac{x}{t}\cdot\frac{x}{t+1}\cdot\frac{x}{t+2}\cdots\frac{x}{n}$$

$$< \frac{x^t}{t!}r\,.\,r\ldots r \text{ to } \overline{n-t} \text{ factors, if } \frac{x}{t+1} = r,$$

for then $$\frac{x}{t+2} < r\ldots\frac{x}{n} < r, \quad \therefore \frac{x^n}{n!} < \frac{x^t}{r^t\,.\,t!}\cdot r^n.$$

Now $1-r = \dfrac{t+1-x}{t+1}$, a proper fraction independent of n, and $\dfrac{x^t}{r^t\,.\,t!}$ is a constant quantity. Hence n can be taken so that $\dfrac{x^n}{n!} < \epsilon$, by the previous paragraphs.

$\underset{\theta\to 0}{\text{L}t}\,\dfrac{\sin\theta}{\theta} = 1$, where θ is the radian measure of an angle, for $\dfrac{\sin\theta}{\theta} = 1 - \kappa_1\frac{1}{4}\theta^2$, where κ_1 is a proper fraction (p. 70); $1 - \dfrac{\sin\theta}{\theta} < \epsilon$, if $\theta^2 < \dfrac{4\epsilon}{\kappa_1}$, which can evidently be obtained.

In a similar way it is easily shown that

$$\underset{\theta\to 0}{\text{L}t}\,\frac{\tan\theta}{\theta} = 1, \quad \underset{\theta\to 0}{\text{L}t}\,\frac{1-\cos\theta}{\theta^2} = \frac{1}{2}.$$

Limit of a Product.

Let l_1, l_2 be limits towards which $f_1(x)$, $f_2(x)$ tend as x approaches the value a.

Then, by definition of a limit, h can be found so that

$$f_1(a+h)-l_1 = \pm\epsilon_1, \text{ and } f_2(a+h)-l_2 = \pm\epsilon_2,$$

where ϵ_1 and ϵ_2 are any small quantities assigned.

Let $\mathbf{F}(\mathbf{x}) = \mathbf{f}_1(\mathbf{x}) \times \mathbf{f}_2(\mathbf{x})$.

Then $F(a+h)-l_1 l_2 = f_1(a+h) \times f_2(a+h) - l_1 l_2$

$$= (l_1 \pm \epsilon_1)(l_2 \pm \epsilon_2) - l_1 l_2 = \pm\epsilon_1 l_2 \pm \epsilon_2 l_1 \pm \epsilon_1 \epsilon_2.$$

Now assign a quantity ϵ, and suppose ϵ_1, ϵ_2 to have been chosen so that $\epsilon_1 < \frac{1}{2l_2}\cdot\epsilon$, and then $\epsilon_2 < \frac{1}{2(l_1+\epsilon_1)}\cdot\epsilon$.

Then $\epsilon_1 l_2 + \epsilon_2(l_1+\epsilon_1) < \epsilon$, and whether the signs of ϵ_1, ϵ_2 are like or unlike, $F(a+h) \backsim l_1 l_2 < \epsilon$;

$$\therefore \underset{\mathbf{x}\to\mathbf{a}}{\mathrm{L}^t}\,\mathbf{F}(\mathbf{x}) = l_1 l_2 = \underset{\mathbf{x}\to\mathbf{a}}{\mathrm{L}^t}\,\mathbf{f}_1(\mathbf{x}) \times \underset{\mathbf{x}\to\mathbf{a}}{\mathrm{L}^t}\,\mathbf{f}_2(\mathbf{x}).$$

It is well to illustrate this theorem numerically.

Consider the quantity $\frac{\sin 2\theta \,.\, \sin 3\theta}{\theta^2}$,

$\sin 2\theta = 2\theta - \kappa_1 \frac{1}{4}(2\theta)^3$, where κ_1 is a proper fraction (p. 70),

$\frac{\sin 2\theta}{\theta} - 2 = -\epsilon_1$ (say), where $\epsilon_1 = 2\kappa_1 \,.\, \theta^2 < 2\theta^2$,

$\frac{\sin 3\theta}{\theta} - 3 = -\epsilon_2$, where $\epsilon_2 < \frac{27}{4}\theta^2$,

$\frac{\sin 2\theta \,.\, \sin 3\theta}{\theta^2} = (2-\epsilon_1)(3-\epsilon_2) = 6 - 3\epsilon_1 - \epsilon_2(2-\epsilon_1)$.

Assign a quantity ϵ, say $\frac{1}{10^6}$. Take $3\epsilon_1 < \frac{1}{2\times 10^6}$; this will be secured if $\theta < \frac{1}{10^3\sqrt{12}}$. Take

$\epsilon_2(2-\epsilon_1) < \frac{1}{2\times 10^6}$, $\epsilon_2 < \frac{1}{4\times 10^6 - 12\times 10^{12}}$; this will be secured if $\theta = \frac{1}{6\times 10^3}$.

Then $6 - \frac{\sin 2\theta \,.\, \sin 3\theta}{\theta^2} < \frac{1}{10^6}$, if $\theta < \frac{1}{6 \times 10^3}$.

$$\mathop{\text{L}^t}_{\theta \to 0} \frac{\sin 2\theta \,.\, \sin 3\theta}{\theta^2} = 2 \times 3 = \mathop{\text{L}^t}_{\theta \to 0} \frac{\sin 2\theta}{\theta} \,.\, \frac{\sin 3\theta}{\theta}.$$

EXAMPLE. Obtain the corresponding theorem when

$$F(x) = f_1(x) \div f_2(x).$$

The theorem can be extended to any finite number (t) of factors, by choosing $\epsilon_1, \epsilon_2, \ldots \epsilon_t$ each $< \frac{\epsilon}{t \,.\, l_t}$, with sufficient allowance for products of the ϵ's.

Hence $\mathop{\text{L}^t}_{x \to a} \{f(x)\}^t = \left\{\mathop{\text{L}^t}_{x \to a} f(x)\right\}^t$, where t is a positive integer.

t cannot be increased indefinitely, for then an ϵ would have to be found for $f(x)$ indefinitely less than the ϵ ultimately assigned.

Limit of a Sum or Difference.

With similar notation, if $F(x) = f_1(x) \pm f_2(x)$, $\text{L}^t f(x)$ differs from $l_1 \pm l_2$ by less than $\epsilon_1 + \epsilon_2$, that is, by less than ϵ, if $\epsilon_1 = \epsilon_2 = \frac{\epsilon}{2}$.

$$\therefore \mathop{\text{L}^t}_{x \to a} F(x) = \mathop{\text{L}^t}_{x \to a} f_1(x) \pm \mathop{\text{L}^t}_{x \to a} f_2(x),$$

and by taking $\epsilon_1 = \epsilon_2 = \ldots \frac{\epsilon}{t}$, the theorem can be extended to any finite number of terms.

EXAMPLES ON LIMITS.

1. Find the variable n or x so that (i) $\epsilon = \frac{1}{100}$, (ii) $\epsilon = \frac{1}{10^6}$. and find the limits of

(a) $\frac{2}{1-(\frac{2}{3})^n} (n \longrightarrow \infty)$, (b) $\frac{3+4x}{7-9x} (x \longrightarrow \infty)$, (c) $\frac{x^3+8}{x+2} (x \longrightarrow -2)$,

(d) $\frac{3x^2+5x}{4x^2-2x} (x \longrightarrow 0$ and $x \longrightarrow \infty)$, (e) $\frac{1-\cos\theta}{\sin\theta \,.\, \sin\frac{\theta}{2}} (\theta \longrightarrow 0)$,

(f) $\sec\theta - \cos\theta \, (\theta \longrightarrow 0)$, (g) $\frac{20^n}{n!} (n \longrightarrow \infty)$.

Also draw the graphs of (c), (e), and (f) in the neighbourhoods of their limiting values.

[As example of p. 103, let us choose n so as to make $\frac{4^n}{n!} < \frac{1}{1000}$.

$$\frac{4^n}{n!} = \frac{4^4}{1.2.3.4} \cdot \frac{4}{5} \cdot \frac{4}{6} \ldots < \frac{32}{3} \cdot \frac{4}{5} \cdot \frac{4}{5} \ldots < \frac{32}{3} \cdot \frac{1}{(1+\frac{1}{4})^{n-4}}$$

$< \frac{32}{3} \cdot \frac{1}{(n-4)\cdot\frac{1}{4}}$ (as on p. 102); and this is

$< \frac{1}{1000}$ if $32000 < \frac{3}{4}n - 3$, if $n > 42672$.*]

2. Find the difference between $\left(1 + \frac{1}{n}\right)^n$ and 2·7183 when $n = 1, 2, 10, 100$, using logarithms.

Series and Convergency.

If $u_1, u_2, u_3, \ldots u_t \ldots$ are algebraic quantities, such that a term u_t is a function of t, the other quantities involved being the same for all terms, then these quantities are said to form a *sequence*, and the expression $u_1 + u_2 + \ldots u_n$ is called a *series*. u_t, expressed as a function of t, is called the *general term*.

Thus the progressions

$$a,\ a+d,\ \ldots\ a+\overline{t-1}\,d,\ \ldots,$$

$$a,\ ar,\ \ldots\ ar^{t-1},\ \ldots, \text{ are sequences;}$$

and $\quad a^n + na^{n-1}x + \ldots + {}_nC_t\, a^{n-t}x^t + \ldots$ is the Binomial Series.

EXAMPLES. 1. Write out the first six terms of each of the series whose general terms are

(i) $\frac{1}{a+(t-1)d}$, (ii) $\frac{(-x)^{t-1}}{(t-1)!}$, (iii) $\sin\left(\alpha + t\cdot\frac{\pi}{2}\right)$, (iv) $t\cdot x^{t-1}$.

2. Write down the general terms of

(i) $1.2 + 2.3 + 3.4 + \ldots,$

(ii) $1.2.4 + 2.3.6x + 3.4.8x^2 + \ldots.$

DEFINITION. Let u_t be any function of t, and let

$$S_n = u_1 + u_2 + \ldots + u_t + \ldots u_n;$$

then if $\underset{n\to\infty}{\mathrm{L}^t}\, S_n$ is zero or finite, and unambiguous, the series

* Note in these and similar cases the object is to show in the easiest way that a value of n can be found which gives the result, not to find the smallest value of n. Here $n = 15$ is in fact sufficient.

$S = u_1 + u_2 + \dots + u_n + \dots$ continued indefinitely is said to be *convergent.*

Thus, if $u_t = ar^t$, where $r = 1 - d$, d being a proper fraction, it is shown, p. 102, that $\underset{n \to \infty}{\text{L}^t} S_n = \frac{a}{1-r}$; $\therefore$ the geometric progression is convergent for such values of r.

If $u_t = (-1)^t$, $S = -1 + 1 - 1 + 1 \dots$, and S_n is 0 or -1 according as n is odd or even; this is ambiguous.

If $u_t = a + (t-1)\, d$, d being positive and finite, then

$$S_n = \frac{n}{2}(2a + \overline{n-1}\, d)$$

and increases indefinitely. Such a series is said to be *divergent.*

Let S be the limit of a convergent series $u_1 + u_2 + \dots$, and let $R_n = u_{n+1} + u_{n+2} + \dots$; then R_n is called the remainder after n terms.

Since $\underset{n \to \infty}{\text{L}^t} S_n = S$, we can by increasing n make the difference between S and S_n, that is R_n, as small as we please.

$$\therefore \underset{n \to \infty}{\text{L}^t} R_n = 0.$$

Conversely, if $\underset{n \to \infty}{\text{L}^t} R_n = 0$, the series is convergent; for S_n differs from its final value by R_n, which approaches zero as n increases; that is, the final value of S_n is between an assigned value $S_{n'}$ and $S_{n'} \pm \epsilon$, where ϵ can be made as small as we please. Hence S_n can be confined within as narrow a margin as we please and has therefore an unambiguous finite limit. We cannot always determine this limit; that is, we cannot always express the sum of the series as a definite quantity, even though we know that the limit exists; but we can obtain as close a numerical approximation as we please.

E.g. consider the series $x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \frac{x^t}{t} + \dots$ where $0 < x < 1$.

$$R_n = \frac{x^{n+1}}{n+1} + \frac{x^{n+2}}{n+2} + \dots < \frac{x^{n+1}}{n+1}(1 + x + x^2 + \dots) < \frac{x^{n+1}}{(n+1)(1-x)}.$$

Take the case where $x = \frac{1}{10}$, $\epsilon = \frac{1}{1000}$.

$$R_n < \epsilon, \text{ if } \left(\frac{1}{10}\right)^{n+1} < (n+1)\frac{9}{10} \cdot \frac{1}{10^3}, \text{ if } 9 > \frac{1}{(n+1) \,.\, 10^{n-3}},$$

if $n > 2$.

Hence S is between $S_2 = \frac{1}{10} + \frac{1}{200} = \cdot 105$ and $S_2 + \frac{1}{1000}$, i.e. between $\cdot 1054$ and $\cdot 1063$.

As close an approximation as we please can be obtained by a similar process.

The ratio test of Convergency.

It is evident that a series consisting of positive terms cannot be convergent if each term is greater than or even equal to the preceding. E.g. $1 + 2 + 3 + 4 + \ldots$.

It is necessary to test the converse, that is, to find whether a series is convergent if each term is less than the preceding.

Dealing only with positive terms, let $\frac{u_{t+1}}{u_t} = r_t$, that is, let $r_1, r_2 \ldots r_t \ldots r_n$ be the ratios of successive terms to those next before them.

Then $u_2 = u_1 r_1$, $u_3 = u_2 r_2 = u_1 r_1 r_2, \ldots$ $u_t = u_1 r_1 r_2 \ldots r_{t-1}, \ldots$

$$u_{n+1} = u_1 r_1 r_2 \ldots r_n.$$

$$R_n = u_{n+1} + u_{n+2} + \ldots$$
$$= u_{n+1}(1 + r_{n+1} + r_{n+1} r_{n+2} + r_{n+1} r_{n+2} r_{n+3} + \ldots).$$

If the *greatest* of the quantities $\mathbf{r_1, r_2 \ldots r_\infty = 1 - d = r}$, where d is a positive proper fraction, independent of n, the series is convergent.

For $R_n < u_{n+1}(1 + r + r^2 + \ldots) < \frac{u_{n+1}}{1-r}$ (p. 102)

$$< \frac{u_1 r_1 r_2 \ldots r_n}{1-r} < \frac{u_1 r^n}{d}.$$

$\therefore \underset{n \to \infty}{\mathrm{L}^t} R_n = \underset{n \to \infty}{\mathrm{L}^t} \frac{u_1}{d} \cdot r^n = 0$, d being finite and independent of n (p. 102).

Thus $1 + x + \frac{x^2}{2!} + \ldots + \frac{x^t}{t!} + \ldots$ is convergent if $0 < x < 1$ and $1 - x$ is finite.

For $r_1 = x$, $r_2 = \frac{x}{2} \ldots$, the greatest ratio is x, which satisfies the condition.

EXAMPLES. 1. In the series just named, find n in terms of x so that $R_n < \frac{1}{10^6}$, and evaluate the series to the 5th decimal place when $x = \frac{1}{10}$.

2. Show that $\frac{x}{1.2}+\frac{x^2}{2.3}+\ldots$ is convergent if $0<x<1$, find n so that $R_n<\frac{1}{10^5}$ when $x=\frac{1}{2}$, and then evaluate the sum of the series to the fourth decimal place.

Extensions of Ratio Test.

I. If the condition $r_t<r$, where $r=1-d$, holds for all the terms except the first s, where s is a finite integer, the series is convergent.

For let $R_s=S'$, $S=S_s+S'$. S_s is finite.

Then S' is convergent; let its limit be l. Then

$$\operatorname*{Lt}_{n\to\infty} S_n=\operatorname{Lt}(S_s+S')=S_s+l,$$

and S is convergent.

Example. $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$ is convergent if x is any finite positive quantity.

For, let $x=s-1$, an integer, or be between $s-1$ and s; then $r_s=\frac{x}{s}$ and is less than 1 by the finite proper fraction $\frac{s-x}{s}$, and

$$r_s>r_{s+1}>r_{s+2}\ldots.$$

II. If $\operatorname*{Lt}_{m\to\infty}\frac{u_{m+1}}{u_m}=1-d$, where d is a positive proper fraction, independent of n, the series is convergent.

For a finite value of m can be found so that $\frac{u_{m+1}}{u_m}=1-d\pm\epsilon_1$, where ϵ_1 is less than as small a quantity as we choose to assign, $=1-d'$, where d' is a positive proper fraction. (E.g. if $d=\frac{1}{10}$, and $\epsilon_1=\frac{1}{100}$, $d'=\frac{9}{100}$.)

From and after the terms so chosen the ratios are all between $1-d'$ and $1-d$, and the series is convergent by the previous case.

Examples. 1. Show that $\frac{1}{2}x+\frac{2}{3}x^2+\ldots+\frac{t}{t+1}x^t+\ldots$ is convergent if x is between 0 and 1.

2. Show $\frac{2}{1}x + \frac{3}{2}x^2 + \ldots + \dfrac{t+1}{t}x^t + \ldots$ is convergent if x is between 0 and 1.

3. In the second example, work through the proof when $x = \frac{9}{10}$, finding d and d' when ϵ is taken as 10^{-6}.

If $S = u_1 + u_2 + \ldots + u_t + \ldots$ is convergent when all the terms are positive, then $S' = u_1 \pm u_2 \pm \ldots \pm \ldots \pm u_t \pm \ldots$ is convergent.

For $R'_n = \pm u_{n+1} \pm u_{n+2} \pm \ldots$ and is between $(+u_{n+1} + u_{n+2} + \ldots)$ and $-(u_{n+1} + u_{n+2} + \ldots)$, i.e. between $\pm R_n$.

But $\underset{n \to \infty}{\text{L}}{}^t R_n$ is 0 by hypothesis; $\therefore \underset{n \to \infty}{\text{L}}{}^t R'_n = 0$ and S' is convergent.

The most useful case is when $S' = u_1 - u_2 + u_3 - \ldots$.

NOTE.—S' may be convergent though S is divergent.

Summarizing the results so far:

If $\pm \underset{m\,\infty}{\text{L}}{}^t \dfrac{u_{m+1}}{u_m} = 1 - d$, where d is a positive proper fraction, S is convergent.

It is easily seen that if $\dfrac{u_{t+1}}{u_t} > 1$ when t is finite, the series is divergent; and it can be deduced that if $\pm \underset{m\,\infty}{\text{L}}{}^t \dfrac{u_{m+1}}{u_m} > 1$ the series is ambiguous or divergent; but we are only concerned here with establishing the *con*vergency of the series we are about to use.

A common mistake is to forget that d must differ finitely from zero. Neglect of this readily leads to absurd results.

E.g. if $u_t = \dfrac{1}{t}$, $S = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{t} + \ldots$

$$\frac{u_{t+1}}{u_t} = \frac{t}{t+1} = 1 - \frac{1}{t+1},$$

which is < 1 if t is finite.

But no quantity can be found so that $\dfrac{u_{t+1}}{u_t}$ is always less than $1 - d$, for $\dfrac{u_{t+1}}{u_t}$ tends to 1 as t is increased. $\text{L}^t \dfrac{u_{t+1}}{u_t} = 1$.

In fact $S = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) +$

$> 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) +$

$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} +$ continued indefinitely.

As an exercise in method the following two theorems are given: If $S = u_1 - u_2 + u_3 - \ldots$, where $u_1 > u_2 > u_3 > \ldots$, all the letters stand for positive quantities and $\underset{m \to \infty}{\mathrm{L}}^{t}\, u_m = 0$, the series is convergent.

For $\quad R_n = (u_{n+1} - u_{n+2}) + (u_{n-3} - u_{n+4}) + \ldots$ and is positive,

$= u_{n+1} - (u_{n+2} - u_{n+3}) - \ldots$ and is $< u_{n+1}$.

$$\therefore \mathrm{L}^{t}\, R_n \not> \mathrm{L}^{t}\, u_{n+1} = 0.$$

Example. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + -$ is convergent.

If $\qquad S = u_1 + u_2 + u_3 + \ldots$

and $\qquad S' = v_1 + v_2 + v_3 + \ldots$

and if $\frac{v_t}{u_t} = k_t$, then if S is convergent and k_t finite for all pairs of terms, and all the terms are positive, S' is convergent.

For $\frac{R'_n}{R_n} = \frac{v_{n+1} + v_{n+2} + \ldots}{u_{n+1} + u_{n+2} + \ldots} <$ the greatest of the ratios $k_{n+1}, k_{n+2} \ldots$

(pp. 17–18.)

Let the greatest ratio, finite by hypothesis, be k

$$R'_n < kR_n.$$

But R_n can be made less than $\epsilon' = \frac{k}{\epsilon}$ where ϵ' and ϵ are as small as we please. $\therefore R'_n$ can be made $< \epsilon$. $\therefore \mathrm{L}^{t}\, R'_n = 0$.

Corollary. S' is convergent if some of its terms are negative and

$$\frac{|v_t|}{v_t} < k.$$

Example. $\frac{1}{a} + \frac{1}{a + r} + \frac{1}{a + r^2} + \frac{1}{a + r^3} +$ is convergent if r is greater than 1. $\left[\text{Take } u_t = \left(\frac{1}{r}\right)^t\right]$.

Applications.

I. Let $u_{t+1} = \frac{[m]_t}{t!} \cdot x^t$, using the notation of p. 19,

and $$S = 1 + mx + \frac{m(m-1)}{2} x^2 + \dots + \frac{[m]_t}{t!} x^t + \dots,$$

where m is any finite quantity.

This we already know as the binomial series when m is a positive integer.

$$\frac{u_{t+1}}{u_t} = \frac{[m]_t (t-1)!}{[m]_{t-1}\, t!} \cdot \frac{x^t}{x^{t-1}} = \frac{m-t+1}{t} \cdot x = -\left(1 - \frac{m+1}{t}\right) x;$$

$$\therefore \quad -\mathop{\mathrm{L}}_{t\,\infty}^{\mathrm{t}} \frac{u_{t+1}}{u_t} = \mathop{\mathrm{L}}_{t\,\infty}^{\mathrm{t}} \left(1 - \frac{m+1}{t}\right) x \; = x, \; m \text{ being finite.}$$

The series is therefore convergent when $|x| < 1$.

If $|x| > 1$ the series is not convergent.

If $x = \pm 1$, we do not know by this method whether the series is convergent or not.

II. $S_1 = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 +$

$$\mathrm{L}^{\mathrm{t}} \frac{u_{t+1}}{u_t} = \mathrm{L}^{\mathrm{t}} \frac{x_{t+1}}{t+1} \cdot \frac{t}{x_t} = \mathrm{L}^{\mathrm{t}}\, x \left(1 - \frac{1}{t+1}\right) = x.$$

Convergent if $|x| < 1$.

Then $S_2 = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$ is convergent if $1 > x > 0$.

If $x = 1$, S_1 is divergent and S_2 is convergent (p. 111).

III. $S = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \dots \; + (-1)^n \frac{\theta^{2n}}{(2n)!} \dots$ (See p. 241.)

$$-\mathrm{L}^{\mathrm{t}} \frac{u_{n+1}}{u_n} = \mathrm{L}^{\mathrm{t}} \frac{\theta^{2n}}{(2n)!} \cdot \frac{(2n-2)!}{\theta^{2n-2}} = \mathrm{L}^{\mathrm{t}} \frac{\theta}{2n} \times \frac{\theta}{2n+1} = 0,$$

whatever finite value θ has, and S is convergent.

Similarly, $\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$ is convergent.

EXAMPLES.

1. Show that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots<\frac{1}{1^2}+2\times\frac{1}{2^2}+4\times\frac{1}{4^2}+8\times\frac{1}{8^2}\ldots$$

and is convergent.

2. Show that $\frac{1}{1^p}+\frac{1}{2^p}+\ldots+\frac{1}{t^p}+\ldots$ is convergent if $p>1$, and, by comparison with $\frac{1}{1}+\frac{1}{2}+\ldots$ (p. 111), is divergent when $p=1$ or $1>p$.

3. If $u_t=\dfrac{1}{a+(t-1)d}$, series is divergent.

4. If $u_t=\dfrac{x^t}{t(t+1)(t+2)}$, series is convergent if $|x| \ngtr 1$.

Multiplication of convergent series.

Let $$S=u_1+u_2+\ldots+u_t+\ldots$$
and $$S'=v_1+v_2+\ldots+v_t+\ldots,$$
both series being convergent and all the terms positive.

Let $$\Sigma=(u_1v_1)+(u_1v_2+u_2v_1)+(u_1v_3+u_2v_2+u_1v_3)+\ldots$$
$$+(u_tv_1+u_{t-1}v_2+u_{t-2}v_3+\ldots+u_1v_t)+\ldots$$
$$=w_1+w_2+w_3+\ldots+w_t+\ldots.$$

[To understand the formation of the w's multiply $u_1+u_2x+u_3x^2+\ldots$ by $v_1+v_2x+v_3x^2+\ldots$ and collect terms in powers of x.]

Write Σ_{2n} for $w_1+w_2+\ldots w_{2n}$, S_n for $u_1+u_2+\ldots u_n$, &c.

Σ_{2n} contains all the terms in the product $S_nS'_n$ and others besides.

$S_{2n}S'_{2n}$ contains all the terms in Σ_{2n} and others besides.

[If $n=1$, $S_2S'_2=u_1v_1+(u_1v_2+u_2v_1)+u_2v_2=\Sigma_2+u_2v_2$,
but $\Sigma_2=S_1S'_1+u_1v_2+u_2v_1$.]

$\therefore$ when n is finite, $S_{2n}S'_{2n}>\Sigma_{2n}>S_nS'_n$.

But S_{2n} and S_n have the same limit S, and S'_{2n}, S'_n have the same limit S';

$$\therefore \Sigma=\underset{n\to\infty}{\mathrm{L}^t}\,\Sigma_{2n}=\mathrm{L}^t\,S_nS'_n$$
$$=\mathrm{L}^t\,S_n\times\mathrm{L}^t\,S'_n \text{ (p. 104)}=S.S'.$$

If some terms are negative, their numerical values being as before, $\Sigma_{2n} - S_n S'_n$ contains the same letters as before, but some terms are negative.

But n can be taken so that $(\Sigma_{2n} - S_n S'_n) < \epsilon$ when all the terms are positive.

$\therefore$ $\Sigma_{2n} - S_n S'_n$ is between $+\epsilon$ and $-\epsilon$, when some are negative (compare p. 110).

$$\therefore \mathop{\mathrm{L}}^{t} (\Sigma_{2n} - S_n S'_n) = 0,$$

and
$$\Sigma = \mathop{\mathrm{L}}^{t} \Sigma_{2n} = \mathop{\mathrm{L}}^{t} S_n S'_n = SS',$$

whether all terms are positive or not.

Lemma.

Vandermonde's theorem.

Let m and n be any integers and t an integer less than either.

Then
$$_{m+n}C_t = {}_mC_t + {}_mC_{t-1} \cdot {}_nC_1 + {}_mC_{t-2} \cdot {}_nC_2 + \dots$$
$$+ {}_mC_{t-s} \cdot {}_nC_s + \dots + {}_nC_t,$$

for the left-hand side is the number of ways t things can be chosen from two groups of m and n things respectively when mixed together, while the $(s+1)^{\text{th}}$ term on the right-hand side is the number of ways they can be chosen if s are taken from the n group, and $(t-s)$ from the m group.

$$\therefore \frac{[m+n]_t}{t!} = \frac{[m]_t}{t!} + \frac{[m]_{t-1}}{(t-1)!} \cdot \frac{n}{1} + \dots + \frac{[m]_{t-s}}{(t-s)!} \cdot \frac{[n]_s}{s!} + \dots.$$

Multiply by $t!$

$$[m+n]_t = [m]_t + t \cdot [m]_{t-1}\, n + \dots + {}_tC_s \cdot [m]_{t-s}\, [n]_s + \dots + [n]_t$$

since ${}_tC_s = \dfrac{t!}{(t-s)!\, s!}$ (p. 20).

$$\Big[\text{e.g. } [m+n]_2 = (m+n)(m+n-1) = m(m-1) + 2mn + n(n-1)$$
$$= [m]_2 + 2\,[m]_1\,[n]_1 + [n]_2$$
$$= m^2 + m(2n-1) + (n^2 - n)$$
$$= n^2 + n(2m-1) + (m^2 - m).\Big]$$

If these terms were multiplied out each would be seen to be a rational integral function of the t^{th} degree in m.

Now this equation is true for all integral values of $m > t$, that is, at any rate, for more than t values. ∴ by p. 88 the equation is true for all values of m. [In other words, the result of removing the brackets and writing out the terms in full is to obtain identical expressions on the two sides.]

Whatever value of m we take, a similar argument shows that the equation is true for all values of n.

∴ the equation is true for all values of m and n, t being any positive integer.

E.g. $\left[\frac{1}{2}-\frac{1}{3}\right]_3 = \left[\frac{1}{6}\right]_3 = \frac{1}{6}\left(-\frac{5}{6}\right)\left(-\frac{11}{6}\right) = \frac{55}{216}$,

and $\left[\frac{1}{2}\right]_3 + 3\left[\frac{1}{2}\right]_2\left[-\frac{1}{3}\right]_1 + 3\left[\frac{1}{2}\right]_1\left[-\frac{1}{3}\right]_2 + \left[-\frac{1}{3}\right]_3$

$$= \tfrac{1}{2}\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right) + 3\left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{3}\right) + 3\left(\tfrac{1}{2}\right)\left(-\tfrac{1}{3}\right)\left(-\tfrac{4}{3}\right)$$
$$+\left(-\tfrac{1}{3}\right)\left(-\tfrac{4}{3}\right)\left(-\tfrac{7}{3}\right) = \tfrac{3}{8} + \tfrac{1}{4} + \tfrac{2}{3} - \tfrac{28}{27}$$
$$= \frac{81+54+144-224}{216} = \frac{55}{216}.$$

Binomial Theorem.

We should not in strict language speak of the *sum* of an infinite series, but only of the limit of the sum when the number of terms is indefinitely increased.

We shall now prove that

$$\mathop{\mathrm{L}^t}_{n\to\infty}\left(1 + mx + \frac{m(m-1)}{1.2}x^2 + \ldots + \frac{[m]_t}{t!}x^t + \frac{[m]_n}{n!}x^n\right)$$

is the real positive value of $(1+x)^m$, when $|x| < 1$, and m is any commensurable quantity, positive or negative.

This is generally, but less accurately, written

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1.2}x^2 + \ldots + \frac{[m]_t}{t!}x^t + \ldots.$$

Take $1 > x > -1$; then the series is convergent (p. 112) for all finite values of m.

Let $f(n)$ be the limit of $1 + nx + \ldots + \dfrac{[n]_t}{t!}x^t + \ldots$, where n has any value.

$$f(m_1) = 1 + m_1 x + \frac{m_1(m_1-1)}{1.2}x^2 + \ldots + \frac{[m_1]_t}{t!}x^t + \ldots,$$

$$f(m_2) = 1 + m_2 x + \frac{m_2(m_2-1)}{1.2}x^2 + \ldots + \frac{[m_2]_t}{t!}x^t + \ldots.$$

Then, by p. 113, $f(m_1) \times f(m_2)$ = limit of series whose $(t+1)^{th}$ term is

$$x^t \left\{ \frac{[m_1]_t}{t!} + \frac{[m_1]_{t-1}}{(t-1)!}\frac{m_2}{1} + \frac{[m_1]_{t-2}}{(t-2)!}\frac{[m_2]_2}{2!} + \dots \right.$$

$$\left. + \frac{[m_1]_{t-s}\,[m_2]_s}{(t-s)!\;s!} + \dots + \frac{[m_2]_t}{t!} \right\}$$

$$= \frac{x^t}{t!}\left\{[m_1]_t + \frac{t}{1}[m_1]_{t-1}[m_1] + \dots + {}_tC_s\,[m_1]_{t-s}\,[m_2]_s + \dots + [m_2]_t\right\},$$

$$= \frac{x^t}{t!}[m_1 + m_2]_t \text{ by p. } 114,$$

since ${}_tC_s = \dfrac{t!}{(t-s)!\;s!}$ (p. 20).

$\therefore\ f(m_1) \times f(m_2)$

$$= \mathrm{L}^t \left\{ 1 + (m_1 + m_2)\,x + \frac{[m_1 + m_2]_2}{1\,.\,2}x^2 + \dots + \frac{[m_1 + m_2]_t}{t!}x^t + \dots \right\}$$

$= f(m_1 + m_2)$.

This is true for all finite values of m_1 and m_2.

It follows at once that

$$f(m_1) \times f(m_2) \times f(m_3) \times\ = f(m_1 + m_2 + m_3 + \dots).$$

Evidently, $f(0) = 1$, $f(1) = 1 + x$, $f(2) = 1 + 2x + x^2$.

CASE I. m a positive integer.

$$(1+x)^m = \{f(1)\}^m = f(1 + \dots \text{ to } m \text{ terms})$$
$$= f(m) = 1 + mx + {}_mC_2\,x^2 + {}_mC_t\,x^t + \dots + x^m. \quad \text{(See p. 23.)}$$

CASE II. m a positive commensurable fraction $= \dfrac{p}{q}$, where p and q are positive integers.

$$\left\{f\left(\frac{p}{q}\right)\right\}^q = f\left(\frac{p}{q}\right) \times f\left(\frac{p}{q}\right) \times \dots q \text{ factors}$$

$$= f\left(\frac{p}{q} + \frac{p}{q} + \dots q \text{ terms}\right) = f\left(\frac{p}{q} \times q\right)$$

$$= f(p) = (1+x)^p \text{ by Case I.}$$

$f\left(\dfrac{p}{q}\right)$ is one of the q^{th} roots of $(1+x)^p$, and, in this case, one of the values of $(1+x)^m = f(m) = 1 + mx + \dots$.

Case III. $m = -k$, where k is a positive integer or positive commensurable fraction.

$f(m) \times f(+k) = f(m+k) = f(0) = 1.$

$$f(m) = \frac{1}{f(k)} = \text{one value of } \frac{1}{(1+x)^k} = \text{one value of } (1+x)^{-k}$$

$$= \text{one value of } (1+x)^m.$$

If m is integral there is of course no ambiquity.

Hence for all commensurable values of m,

$$\text{L}^t f(m) = \text{one value of } (1+x)^m \text{ when } m \text{ is fractional.}$$

$$= (1+x)^m \text{ when } m \text{ is integral.}$$

The complete proof that the real positive value is always to be taken is difficult. Its nature can, however, be shown, and a partial proof given as follows.

$f(m)$ is evidently real if m and x are real. It remains to show that it cannot be negative.

If q is odd, there is only one real value and that is positive.

If q is even, there are two equal real values of opposite signs.

[See p. 3, also De Moivre's Theorem, p. 231.]

Consider $(1+x)^{\frac{17}{19}}$, $(1+x)^{\frac{17}{20}}$, $(1+x)^{\frac{17}{21}}$, $(1+x)^{\frac{17}{22}}$.

$f(\frac{17}{19})$ and $f(\frac{17}{21})$ are real and positive. $f(\frac{17}{20})$ is formed in exactly the same way as these. It is inconceivable that as n increases through these values (or nearer values, as $\frac{5}{2001}$, $\frac{5}{2002}$, $\frac{5}{2003}$) $f(n)$ should jump from a positive to a nearly equal negative value and back again.

Hence the limit of the series is in all eases positive as well as real, and gives the unique real positive value of $(1+x)^m$.

Expansions of importance. $0 < x < 1$.

$$(1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})x^2}{1\,.\,2} + \ldots = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 \ldots$$

$$(1+x)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \tfrac{35}{128}x^4 \ldots$$

$$(1-x)^{\frac{1}{2}} = 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 \ldots$$

$$(1-x)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \tfrac{5}{16}x^3 + \tfrac{35}{128}x^4 \ldots$$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$ [A geometric progression.]

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$ [A geometric progression.]

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots + (-1)^t(t+1)x^t + \dots$$

$$(1-x)^{-3} = 1 + 3x + 6x^2 + \dots + \frac{(t+1)(t+2)}{2}x^t + \dots.$$

Use in approximation.

$$\left(1 + \frac{1}{10}\right)^{\frac{1}{2}} = 1 + \frac{1}{2} \text{ of } \frac{1}{10} - \frac{1}{8} \text{ of } \frac{1}{10^2} + \frac{1}{16} \cdot \frac{1}{10^3} \cdots$$

$$= 1 + \cdot05 - \cdot00125 + \cdot0000625.$$

1st approx. = 1·05, 2nd 1·04875, 3rd 1·0488125.

From p. 111 these approximations are alternately greater and less than $(1 + \frac{1}{10})^{\frac{1}{2}}$.

$\therefore \sqrt{\frac{11}{10}}$ is between 1·04875 and 1·04881,

$$(1 - \tfrac{1}{10})^{\frac{1}{2}} = 1 - \cdot05 - \cdot00125 + R_3,$$

$$-R_3 = \frac{5}{16} \cdot \frac{1}{10^3} + \frac{5}{16} \cdot \frac{7}{8} \cdot \frac{1}{10^4} + \frac{5}{16} \cdot \frac{7}{8} \cdot \frac{9}{10} \cdot \frac{1}{10^5} + \dots$$

$$< \frac{5}{16} \cdot \frac{1}{10^3}\left(1 + \frac{1}{10} + \frac{1}{10_2} + \dots\right) < \frac{5}{16 \times 900} < \cdot00035,$$

$\therefore \sqrt{\frac{9}{10}} < \cdot94875$ and > 94840.

Examples.

1. Show $(1-x)^{-\frac{1}{5}} = 1 + \frac{1}{5}x + \frac{1 \,.\, 6}{5 \,.\, 10}x^2 + \frac{1 \,.\, 6 \,.\, 11}{5 \,.\, 10 \,.\, 15}x^3 + \dots$

2. Show $(a+x)^n = a^n + na^{n-1}x +$, when $a > x$, and write down the general term.

3. Approximate to $\sqrt{3}$ by expanding $(1 - \frac{1}{4})^{\frac{1}{2}}$.

4. Find a limit for the error made if $\dfrac{a(1+d_1)^2}{b(1-d_2)^3}$ is taken as $\dfrac{a}{b}(1 + 2d_1 + 3d_2)$ when d_1 and d_2 are each less than $\frac{1}{10}$.

Show that $1 + md_1 - nd_2$ is a fair approximation for $\dfrac{(1+d_1)^m}{(1+d_2)^n}$ if d_1 and d_2 are small, and write an expression for the superior limit of the error involved, by considering the remainder after two terms.

5. Expand $\dfrac{(a+x)^{\frac{1}{2}} - a^{\frac{1}{2}}}{x}$ in powers of x, and show that its limit when $x = 0$ is $\dfrac{1}{2\sqrt{a}}$.

The Exponential Series.

The limit of the series $1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \ldots + \dfrac{1}{t!} + \ldots$ is written e, and that of $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots + \dfrac{x^t}{t!} + \ldots$ is written $E(x)$.

This series is convergent for all real values of x (pp. 109, 110). $E(1) = e$.

Value of e.

As on p. 108, if $E(x) = S_n + R_n$, x being positive,

$$R_n = \frac{x^n}{n!}\left(1 + \frac{x}{n+1} + \frac{x}{n+1} \cdot \frac{x}{n+2} + \ldots\right)$$

$$< \frac{x^n}{n!}\left(1 + \frac{x}{n+1} + \left(\frac{x}{n+1}\right)^2 + \ldots\right)$$

$$< \frac{x^n}{n!} \cdot \frac{1}{1 - \dfrac{x}{n+1}}, \text{ when } n + 1 > x.$$

When $x = 1$, $R_n < \dfrac{1}{n!} \cdot \dfrac{1}{1 - \dfrac{1}{n+1}} < \dfrac{n+1}{n \, . \, n!}$.

e can then be evaluated rapidly to any required degree of accuracy, thus:

$$R_{11} < \frac{12}{11!\,11} < \cdot 00000003.$$

The sum of the first 11 terms will then be correct for e to seven decimal places.

In finding S_{11} it is convenient first to find S_7, which gives an easy recurring decimal: then to take two values for S_{11} slightly in defect, and in excess of the true value, thus:

1	$S_{11} >$ 2·71805555	$<$	2·71805556
1·	00019841		00019842
·5	2480		2481
$\cdot 16666\dot{6}$	275		276
$\cdot 04166\dot{6}$	27		28
$\cdot 00833\dot{3}$	2·71828178		2·71828183
$\cdot 00138\dot{8}$			
$S_7 = 2{\cdot}7180555\dot{5}$			

In the middle column each term is too small, in the last column too great.

$$e = S_{11} + R_{11} < 2{\cdot}71828186$$
$$> S_{11} \qquad > 2{\cdot}71828178$$

Similarly, to ten digits e is found to be 2·718281828.

By writing $e = 1 + 1 + \frac{1}{2}\left[1 + \frac{1}{3}\left\{1 + \frac{1}{4}\left(1 + \frac{1}{5}\,\overline{1 +}\right)\right\}\right]$ it can be seen that no fraction $\frac{p}{q}$, p and q integral, can be found to represent e. For if we multiply by q we still have a fractional value for qe, whatever integer q is.

e is therefore incommensurable.

e is, after π, the most important constant in practical mathematics.

Exponential Theorem.*

$$\underset{n \to \infty}{\mathrm{L}}{}^{t}\left(1 + x + \frac{x^2}{2!} + \dots + \frac{x^t}{t!} + \dots \frac{x^n}{n!}\right)$$

equals the real positive value of e^x, when x is commensurable.

$$E(x_1) = 1 + x_1 + \frac{x_1^{\,2}}{2!} + \dots + \frac{x_1^{\,t}}{t!} + \dots,$$

$$E(x_2) = 1 + x_2 + \frac{x_2^{\,2}}{2!} + \dots + \frac{x_2^{\,t}}{t!} + \dots.$$

The product $E(x_1) \times E(x_2) =$ (by p. 113)

$$\mathrm{L}^{t}\Big\{1 + (x_1 + x_2) + \dots$$
$$+ \left(\frac{x_1^{\,t}}{t!} + \frac{x_1^{\,t-1} x^2}{(t-1)!\,.\,1} + \dots + \frac{x_1^{\,t-s}}{(t-s)!} \cdot \frac{x_2^{\,s}}{s!} + \dots + \frac{x_2^{\,t}}{t!}\right) + \dots\Big\}$$

for all values of x_1 and x_2.

* The word *exponent* is used as equivalent to *index*.

The $\overline{t+1}^{\text{th}}$ term is

$$\frac{1}{t!}\left(x_1^t + t\,.\,x_1^{t-1}\,.\,x_2 + \ldots + \frac{t!}{(t-s)!\;s!}\,x_1^{t-s}\,.\,x_2^s + \ldots + x_2^t\right)$$

$= \dfrac{(x_1+x_2)^t}{t!}$ by the Binomial Theorem for a positive integer.

$$\therefore\; E(x_1) \times E(x_2) = \text{limit of } 1 + (x_1+x_2) + \ldots + \frac{(x_1+x_2)^t}{t!} + \ldots$$

$$= E(x_1+x_2).$$

It readily follows that $E(x_1) \times E(x_2) \times E(x_3) \times \ldots$

$$= E(x_1+x_2+x_3+\ldots).$$

Case I. When x is a positive integer $= n$.

$$e^n = \{E(1)\}^2 = E(1) \times E(1) \times \ldots \text{ to } n \text{ factors}$$

$$= E(1+1+\ldots n \text{ terms}) = E(n).$$

Case II. When $x = \dfrac{p}{q}$, p and q being positive integers.

$$\left\{E\left(\frac{p}{q}\right)\right\}^q = E\left(\frac{p}{q}\right) \times E\left(\frac{p}{q}\right) \times \ldots q \text{ factors}$$

$$= E\left(\frac{p}{q} + \frac{p}{q} + \ldots q \text{ terms}\right) = E\left(\frac{p}{q} \times q\right)$$

$$= E(p) = e^p, \text{ by Case I.}$$

$E\left(\dfrac{p}{q}\right)$ is evidently real and positive.

$\therefore\; E\left(\dfrac{p}{q}\right)$ is the real positive q^{th} root of e^p.

Case III. When $x = -x'$, where x' is a positive integer or commensurable fraction.

$$E(x) \times E(x') = E(x+x') = E(0) = 1,$$

$$E(x) = \frac{1}{E(x')} = \frac{1}{e^{x'}} \text{(by Cases I and II)} = e^{-x'} = e^x.$$

$E(x)$ is always positive, since $e^{x'}$ is positive and their product is 1.

$\therefore$ the real positive value of $e^x = E(x)$ for all commensurable values of x, positive or negative.

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But $\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{t-1}{n}\right)$

$$>1-\frac{1+2+\ldots+(t-1)}{n}\text{ (p. 13) }>1-\frac{t(t-1)}{2n}\text{ (p. 14)}.$$

$$\therefore \left(1+\frac{x}{n}\right)^n > 1+x+\frac{1-\frac{1}{n}}{1.2}x^2+\frac{1-\frac{2.3}{2n}}{1.2.3}x^3+\frac{1-\frac{3.4}{2n}}{1.2.3.4}x^4+\ldots$$

$$+\frac{1-\frac{t(t-1)}{2n}}{t!}x^t+\ldots \text{ to } \overline{n+1} \text{ terms},$$

$$>1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}$$

$$-\frac{x^2}{2n}\left(1+x+\frac{x^2}{2!}+\ldots+\frac{x^{t-2}}{(t-2)!}+\ldots \text{ to } \overline{n-1} \text{ terms}\right),$$

$> F_n-\frac{x^2}{2n}F_{n-2}$, where $F_s=1+x+\frac{x^2}{2!}+\ldots+\frac{x^s}{s!}$ for any integral value of s.

Now $0<F_{n-2}<e^x$.

$\therefore F_n-\left(1+\frac{x}{n}\right)^n<\frac{x^2}{2n}F_{n-2}<\frac{x^2}{2n}e^x$, when n is finite.

$$\therefore \underset{n\to\infty}{\mathrm{L}^t}\{F_n\}-\underset{n\to\infty}{\mathrm{L}^t}\left(1+\frac{x}{n}\right)^n \not> \underset{n\to\infty}{\mathrm{L}^t}\frac{x^2}{2n}e^x \not> 0.$$

$\therefore \underset{n\to\infty}{\mathrm{L}^t}\left(1+\frac{x}{n}\right)^n=\underset{n\to\infty}{\mathrm{L}^t}(F_n)=e^x$, when x is positive.

In particular, $\underset{n\to\infty}{\mathrm{L}^t}\left(1+\frac{1}{n}\right)^n=e$.

The proof can be modified to give the limit of $\left(1-\frac{x}{n}\right)^n$.

Thus $\frac{\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{t-1}{n}\right)}{t!}x^t \backsim \frac{x^t}{t!}<\frac{x^t}{2n(t-2)!}$, as above;

$\therefore$ the binomial expansion of $\left(1-\frac{x}{n}\right)^n$ differs term by term, after the second term, from $1-x+\ldots+(-1)^n\frac{x^n}{n!}$ by not more than $\frac{x^3}{2n}$, $\frac{x^4}{2n \cdot 2!}$ ..., and the aggregate difference is *a fortiori* less than the sum of these terms, i.e. less than $\frac{x^2}{2n} F_{n-2}$;

$$\therefore \underset{n \to \infty}{\mathrm{L}^t} \left(1 - \frac{x}{n}\right)^n - \underset{n \to \infty}{\mathrm{L}^t} \left(1 - x + \ldots + (-1)^n \frac{x^n}{n!}\right)$$
$$\ngtr \mathrm{L}^t \frac{x^2}{2n} F_{n-2} \ngtr 0.$$

$$\therefore \underset{n \to \infty}{\mathrm{L}^t} \left(1 - \frac{x}{n}\right)^n = e^{-x}.$$

The limit just obtained has an intimate and important relation to continuous growth by equal relative increments. Thus suppose compound interest to be reckoned at $\frac{r}{n}$ per cent. every n^{th} part of a year on capital £C.

The amount at the end of the year is £$C\left(1 + \frac{r}{100n}\right)^n =$ £$Ce^{\frac{r}{100}}$, when n is indefinitely increased. This is equivalent to a single increment of r_1 per cent. reckoned at the end of the year if

$$Ce^{\frac{r}{100}} = C\left(1 + \frac{r_1}{100}\right); \text{ i.e. if } r = 100 \log_e\left(1 + \frac{r_1}{100}\right).$$

(E.g. if $r_1 = 4$, $r = 3{\cdot}922$).

If, then, the aggregate growth in a finite time is known, the equivalent rate for continuous growth can be found.

This method is the basis of actuarial calculations.

This conception is closely connected with the use of e as the base of (natural) logarithms.

Example. The population of England was enumerated as 30,807,310 on April 1, 1901, and as 34,043,076 on April 3, 1911. Find the population at any intermediate date, assuming continuous growth at a constant relative rate.

[Let n be the number of days from census to census, m from 1st census to date required, P_1, P_2, and P the given and required populations. Then $P_1\left(1 + \frac{r}{n}\right)^n = P_2$, $P_1\left(1 + \frac{r}{n}\right)^m = P$. Regarding a day as infinitesimal, $P_1 e^r = P_2$, $P_1 e^{r\frac{m}{n}} = P$. $\therefore$ $n \log P = (n-m)\log P_1 + m \log P_2$.]

Lemma.

If S is the limit of a series $u_0 + u_1 x + u_2 x^2 + \ldots$, which is known to be convergent when $c > x \not< 0$, where c is some known positive quantity, then $\underset{x \to 0}{\mathrm{L}^t} \dfrac{S - u_0}{x} = u_1$.

For $\dfrac{S-u_0}{x} = u_1 + x(u_2 + u_3 x + u_4 x^2 + \ldots) = u_1 + x \,.\, F$, say.

F is finite for all values of x for which the series is convergent, for S_1, u_0, and u_1 are finite. (F is not independent of x).

If $xF < \epsilon$, i.e. if $x < \dfrac{\epsilon}{F}$, which can always be secured, then

$$\frac{S-u_0}{x} - u_1 < \epsilon.$$

$$\therefore \underset{x \to 0}{\mathrm{L}^t} \frac{S-u_0}{x} = u_1.$$

This method is of frequent application.

In particular, $\underset{\mathbf{x} \to 0}{\mathrm{L}^t} \dfrac{\mathbf{a^x} - 1}{\mathbf{x}} = \log_e \mathbf{a}$, from the last paragraph.

The logarithmic series.

Now write m for x and $1 + x$ for a in the last line, and we have

$$\log_e(1+x) = \underset{m \to 0}{\mathrm{L}^t} \frac{(1+x)^m - 1}{m}$$

$$= \underset{m \to 0}{\mathrm{L}^t}\, x + \frac{m-1}{2}x^2 + \frac{(m-1)(m-2)}{1\,.\,2\,.\,3}x^3 + \ldots + \frac{[m-1]_{n-1}}{n\,!}x^n + R_n,$$

by the binomial series, if we take $|x| < 1$.

Put $m = 0$ in first n terms, n finite; then

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots - (-1)^n \frac{x^n}{n} + \underset{m \to 0}{\mathrm{L}^t} R_n.$$

R_n contains the factor x^{n+1} and the other factor is finite.

$\therefore$ n can be taken so large that $R_n < \epsilon$ whatever m is.

When m tends to zero, R_n remains $< \epsilon$, and

$$\log_e(1+x) = \underset{n\to\infty}{\mathrm{Lt}}\left\{x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 \ldots - (-1)^n . \frac{1}{n}x^n\right\},$$

and writing $-x$ for x,

$$\log_e(1-x) = \underset{n\to\infty}{\mathrm{Lt}}\left\{-x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \ldots \frac{1}{n}x^n\right\}.$$

when $1 > x > 0$.

These series were shown to be convergent on p. 112.

Subtracting the second series from the first we have

$$\log_e \frac{1+x}{1-x} = 2\left(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \ldots\right), \text{ when } x < 1 \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Write $\dfrac{1}{2n+1}$ for x, and we have

$$\log_e \frac{n+1}{n} = \log_e \frac{1 + \dfrac{1}{2n+1}}{1 - \dfrac{1}{2n+1}}, \text{ identically,}$$

$$= 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\right\} \quad . \quad . \quad \text{(ii)}$$

Any of these series can be used for the numerical calculation of logarithms; the last is the most convenient after the logarithm of any integer has been obtained. The following illustrate the method:

$$\log_e 2 = \log_e \frac{1+\frac{1}{3}}{1-\frac{1}{3}} = 2\left\{\frac{1}{3} + \frac{1}{3^4} + \frac{1}{5 . 3^5} + \frac{1}{7 . 3^7} + \ldots\right\} \text{ from (i)}$$

$> \cdot 69312$ (approx. sum of 1st 4 terms)

$< \cdot 69314 + \dfrac{2}{9 . 3^9}\left(\dfrac{1}{1-\frac{1}{9}}\right)$, using remainder after 4 terms

$< \cdot 69316.$

$$\log_e 3 = \log_e 2 + 2\left\{\frac{1}{5} + \frac{1}{3 . 5^3} + \frac{1}{5 . 5^5} + \ldots\right\} \text{ from (ii)}$$

$$= 1\cdot 0986 \text{ approx.}$$

$$\log_e 4 = 2\log_e 2 = 1{\cdot}3863 \text{ approx.}$$

$$\log_e 5 = \log_e 4 + 2\left\{\frac{1}{9} + \frac{1}{3\,.\,9^3} + \ldots\right\} \text{ from (ii)}$$

$$= 1{\cdot}6094 \text{ approx.}$$

$$\log_e 6 = \log_e 2 + \log_e 3 = 1{\cdot}7918 \text{ approx.}$$

$$\log_e 10 = \log_e 2 + \log_e 5 = 2{\cdot}3026 = 2{\cdot}3025851 \text{ more exactly.}$$

$$\log_{10} e = \frac{1}{\log_e 10} = {\cdot}43429448 \text{ very nearly (p. 8, iv, \&c.).}$$

Then $\log_{10} 2 = \log_{10} e \times \log_e 2 = {\cdot}30103\ldots$

$\log_{10} 3 = \log_{10} e \times \log_e 3 = {\cdot}47712\ldots$, &c.

By such methods the logarithms of any numbers to base e (called *natural* or *Napierian logarithms*) can be found and those to base 10 (common logarithms) deduced.

It is to be noticed that the formulae on which logarithmic tables (the most important of all aids to practical computation) depend, involve nearly all the more delicate parts of theoretical analysis so far considered.

Examples. 1. Obtain the natural logarithms of 1, 2, 3, 7, 10, correct to 6 decimal places. Deduce those of 4, 5, 6, 8, 9, 11; and hence obtain the common logarithms of 1, 2, 3... 12.

2. Show that $\log_e \frac{1+m}{1-m} = u + \frac{u^3}{3} + \frac{u^5}{5} + \ldots$ where $u = \frac{2m}{1+m^2}$.

An Important Group of Limits.

We can now establish the following limits:

$$\underset{h\to 0}{\mathrm{Lt}} \frac{(x+h)^n - x^n}{h} = \mathrm{Lt} \frac{x^n}{h}\left\{\left(1+\frac{h}{x}\right)^n - 1\right\}$$

$$= \mathrm{Lt} \frac{x^n}{h} \cdot \left(n\frac{h}{x} + \frac{h^2}{x^2}F\right),$$

by the Binomial Theorem where F is finite, $= \mathrm{n}\mathrm{x}^{\mathrm{n}-1}$, where x is any real quantity and n is commensurable. [Compare p. 126.]

$$\underset{h\to 0}{\mathrm{Lt}} \frac{a^{x+h} - a^x}{h} = \underset{h\to 0}{\mathrm{Lt}} \frac{a^x(a^h - 1)}{h} = \mathrm{a}^{\mathrm{x}}.\log_{\mathrm{e}} \mathrm{a} \quad \text{(p. 123).}$$

In particular, $\underset{h\to 0}{\mathrm{Lt}} \frac{e^{x+h} - e^x}{h} = \mathrm{e}^{\mathrm{x}}$.

$$\operatorname*{Lt}_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h}=\log_a e\operatorname{Lt}\frac{\log_e\left(1+\frac{h}{x}\right)}{h}$$

$$=\log_a e\operatorname{Lt}\frac{1}{h}\left(\frac{h}{x}-\frac{1}{2}\frac{h^2}{x^2}+\ldots\right)=\log_a e\operatorname{Lt}\left(\frac{1}{x}-hF\right)=\frac{\log_a e}{x}.$$

In particular, $\operatorname*{Lt}_{h\to 0}\frac{\log_e(x+h)-\log_e x}{h}=$ . . . . $\frac{1}{x}$.

$$\operatorname*{Lt}_{h\to 0}\frac{\sin(\theta+h)-\sin\theta}{h}=\operatorname{Lt}\frac{2\cos\left(\theta+\frac{h}{2}\right)\sin\frac{h}{2}}{h}\quad\text{(p. 57)},$$

$$=\operatorname{Lt}\cos\left(\theta+\frac{h}{2}\right)\times\operatorname{Lt}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\quad\text{(p. 104)},\ =\ .\ .\ \cos\theta.$$

Similarly, $\operatorname*{Lt}_{h\to 0}\frac{\cos(\theta+h)-\cos\theta}{h}=$ . . . . . $-\sin\theta$.

$$\operatorname*{Lt}_{h\to 0}\frac{\tan(\theta+h)-\tan\theta}{h}=\operatorname{Lt}\frac{\sin(\theta+h-\theta)}{h\cos(\theta+h)\cos\theta}\quad\text{(p. 57)},$$

$$=\operatorname{Lt}\frac{\sin h}{h}\times\frac{1}{\cos(\theta+h)}\times\frac{1}{\cos\theta}=1\times\frac{1}{\cos^2\theta}=\ .\ \sec^2\theta.$$

EXAMPLES. Obtain the limits in the case of sine and cosine from the formulae of p. 70. Find a corresponding limit for cot θ.

SECTION VI

PLANE CO-ORDINATE GEOMETRY

Introduction.

THE method of determining a point P in a figure by the quantities x and y, where x is a distance measured from a fixed point O along a fixed axis OX, and y is the distance MP, perpendicular to OX, has been used again and again in the representation of functions. Hitherto, the method has been used for dealing with $y = f(x)$ when y is an explicit function of x (see pp. 72 seq.), and we have been concerned chiefly with the numerical values of y as x varies, and with the numerical values of x which result in assigned values of y. We have not dealt with the geometrical properties of the graphs obtained.

In this section we deal with the geometrical properties of lines (straight or curved) which are the loci of a point P, whose co-ordinates are x, y, where x and y are connected by any equation; y no longer being necessarily an explicit function of x, but x and y being of similar significance and equal importance. To preserve the geometrical properties the units of length on the axes OX and OY must be the same. The diagrams in Section IV, especially that on p. 85, which represents the relation $3x^2 - 2xy - 12x + 10y + 35 = 0$, should be consulted.

The section on Projection (pp. 36–8) shows how to pass from algebraic to geometrical properties.

Co-ordinates and Points.

Let P be any point in a plane, let OX, OY be two fixed lines at right angles, on which scales (with the same unit) are supposed to be marked from $+\infty$ to $-\infty$. O is called the *origin.*

Draw PM perpendicular to OX, and PL perpendicular to OY.

Then the lengths OM, OL, as read on the scales, are called the *abscissa* and *ordinate* (together the *co-ordinates*) of P, and P is spoken of as the point 'x, y', where $OM = x$, $OL = y$. In the figure P is $(4, 5)$, P_1 $(-3, 6)$, P_2 $(-5, -3)$, and P_3 $(2, -4)$.

If now P is any point in the plane (x_1, y_1) and Q any other

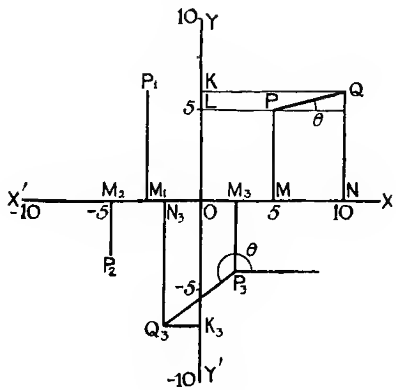


FIG. 48.

point (x_2, y_2), and if NQ, KQ are the co-ordinates of Q, we have for all positions the following relations:

$$x_2 - x_1 = ON - OM = MO + ON = MN;$$
$$y_2 - y_1 = OK - OL = LO + OK = LK.$$

[See MN and $M_3 N_3$ in the figure, and work through the statement numerically.]

MN, LK are the projections of PQ (the line drawn from P to Q, not from Q to P) on OX and OY respectively.

Let PQ make $\angle\, \theta$ with OX, measured positively.

$$\therefore\; PQ \cos\theta = MN = x_2 - x_1,$$

and $$PQ \sin\theta = LK = y_2 - y_1. \quad \text{(p. 48.)}$$

$$\therefore\; \mathbf{PQ}^2 = PQ^2(\cos^2\theta + \sin^2\theta) = (\mathbf{x}_2 - \mathbf{x}_1)^2 + (\mathbf{y}_2 - \mathbf{y}_1)^2. \quad \text{(A) (i)}$$

This formula, giving the distance between two points whose co-ordinates are known, has now been proved for all positions of P and Q.

Division of a Line.

Let R be a point (x', y') in PQ such that $\frac{PR}{RQ} = \frac{m}{n}$. If R is between P and Q, m and n are of the same sign. If R is not

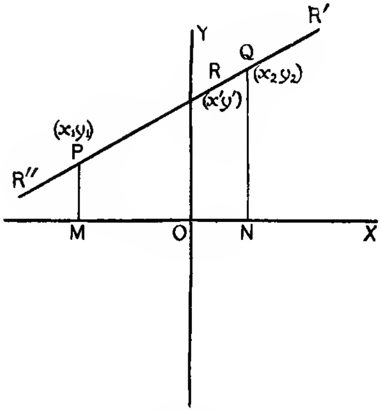


FIG. 49.

between P and Q but has such a position as R' or R'', m and n are of different signs.

In any case $\quad nPR = mRQ,$

$$nPR + mQR = 0. \quad \ldots \ldots \quad \text{(ii)}$$

Let PQ make $\angle\theta$ with OX as before.

Then $PR \cos\theta = x' - x_1$, &c., as above.

Multiply equation (ii) by $\cos\theta$,

$$n(x' - x_1) + m(x' - x_2) = 0, \qquad \mathbf{x}' = \frac{\mathbf{nx_1 + mx_2}}{\mathbf{m + n}}$$

Multiply equation (ii) by $\sin\theta$,

$$n(y' - y_1) + m(y' - y_2) = 0, \qquad \mathbf{y}' = \frac{\mathbf{ny_1 + my_2}}{\mathbf{m + n}} \quad \Bigg\} \text{(B) (iii)}$$

EXAMPLES. If (x_1, y_1), (x_2, y_2), (x_3, y_3) are the angular points of a triangle ABC, and D, E, F the middle points of BC, CA, AB, then the co-ordinates of D are, from (iii),

$$\frac{1 \cdot x_2 + 1 \cdot x_3}{1 + 1} = \frac{x_2 + x_3}{2} \text{ and } \frac{y_2 + y_3}{2} \quad \ldots \ldots \quad \text{(iv)}$$

If G is taken in AD so that $AG = 2GD$, the abscissa of G is, from (iii), $\frac{1 \cdot x_1 + 2 \cdot \frac{1}{2}(x_2+x_3)}{1+2} = \frac{x_1+x_2+x_3}{3}$ and the ordinate $\frac{y_1+y_2+y_3}{3}$.

From the symmetry of the result, G is also a point of trisection of BE and CF.

$$AD^2 = \{x_1 - \tfrac{1}{2}(x_2+x_3)\}^2 + \{y_1 - \tfrac{1}{2}(y_2+y_3)\}^2, \text{ from (i) and (iv).}$$

$$\therefore\ AD^2 + BE^2 + CF^2$$
$$= \tfrac{1}{4}\{(2x_1 - x_2 - x_3)^2 + \dots + \} + \tfrac{1}{4}\{(2y_1 - y_2 - y_3)^2 + \dots + \},$$
$$= \tfrac{1}{4}\{6(x_1{}^2 + \dots +) - 6(x_1 x_2 + \dots + \dots)\} + \text{expression in } y,$$
$$= \tfrac{3}{4}\{(x_1-x_2)^2 + \dots + \dots\} + \text{ expression in } y.$$
$$\therefore\ 4(AD^2 + BE^2 + CF^2) = 3\{(x_1-x_2)^2 + (y_1-y_2)^2\} + \dots + \dots$$
$$= 3(AB^2 + BC^2 + CA^2).$$

EXAMPLE. If E, F, G, H are the middle points of the sides of AB, BC, CD, DA, a quadrilateral, then one point P is the middle point of EG, of FH, and of MN, where M, N are the middle points of AC and BD.

THE EQUATION OF THE FIRST DEGREE, OR THE LINEAR EQUATION.

$\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$ is the most general relation of the first degree between x and y, where A, B, C are constants.

This may be written $y = -\frac{\mathrm{A}}{\mathrm{B}}x - \frac{\mathrm{C}}{\mathrm{B}}$ (unless $\mathrm{B} = 0$) $= mx + k$ $= x\tan\theta + k$, where $m = -\frac{\mathrm{A}}{\mathrm{B}}$, $k = -\frac{\mathrm{C}}{\mathrm{B}}$, and θ is the positive angle whose tangent is m. Since the tangent of an angle is capable of all values, θ can always be found. If $\mathrm{B} = 0$, we have $x = -\frac{\mathrm{C}}{\mathrm{A}}$, the equation of a line parallel to OY, corresponding to $\theta = \frac{1}{2}\pi$ when m is infinite.

If A is 0, we have $\tan\theta = 0$, and the line is $y = -\frac{\mathrm{C}}{\mathrm{B}}$ parallel to OX.

[If A, B become smaller and ultimately vanish, possible values of x or y or both become greater and ultimately infinite.]

Cut off OD on $OY = k$, whether k is positive (Fig. 50 a) or negative (Fig. 50 b).

Through D draw a line making the angle θ with the direction OX.

Let P, any point on this line, have co-ordinates $x_1 = ON$, $y_1 = OL$.

In all positions, using the convention of directions of lines, $y_1 = OL = NP = NM + MP = OD + MP$, where DM meets NP in M.

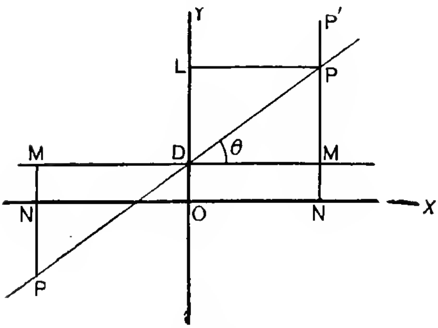


FIG. 50 *a*.

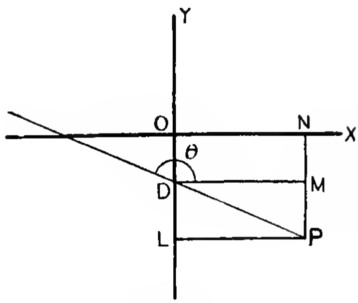


FIG. 50 *b*.

$\therefore\ y_1 - k = MP = DL.$

Regarding D as the trigonometrical origin, and DM as the initial line, $\tan\theta = \dfrac{DL}{DM}$ (p. 43) $= \dfrac{DL}{ON} = \dfrac{y_1 - k}{x_1}$,

$$\therefore\ y_1 = mx_1 + k.$$

$\therefore$ The co-ordinates of any point P on the line through D making the angle θ satisfy the relation $y = mx + k$.

x, y in the equation, co-ordinates of any point that satisfy it, are called *current co-ordinates*; whereas x_1, y_1 are the co-ordinates of an assigned point.

Conversely, if x_1, y_1 co-ordinates of P' satisfy the equation, P' is on the line through D making the angle θ.

For if not, let $x_1 = ON$ and $y_1 = NP'$, and let NP' meet the line at P.

Then $y_1 = mx_1 + k$ by hypothesis.

$$\frac{MP}{DM} = \tan\theta = \frac{y_1 - k}{x_1} = \frac{NP' - OD}{ON} = \frac{NP' - NM}{ON} = \frac{MP'}{DM}.$$

$\therefore\ MP = MP'$, and P' coincides with P, a point on the line.

Hence $Ax + By + C = 0$ is the locus of a point which moves on the straight line which cuts OY at $-\frac{C}{B}$ and makes the angle $\tan^{-1}\frac{A}{B}$ with OX.

$Ax + By + C = 0$ is said to be the equation of this line, and to represent this line.

Thus $x - \sqrt{3}y + 2\sqrt{3} = 0$ or $y = x \tan 30° + 2$ is line (i) in Figure 51.

$4x + 2y + 8 = 0$ or $y = -2x - 4 = x \tan 116\frac{1}{2}° - 4$ is (ii).

$Ax + By + C$ meets OX, where $y = 0$ at $x = -\frac{C}{A} = a$ (say),

and OY, where $x = 0$ at $y = -\frac{C}{B} = b$ (say).

It may be written $\dfrac{x}{-\frac{C}{A}} + \dfrac{y}{-\frac{C}{B}} = 1$, i.e. $\frac{x}{a} + \frac{y}{b} = 1$, and

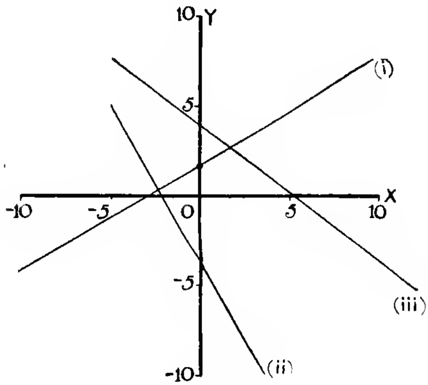


FIG. 51.

indeed it is evident that the equation last written is satisfied by $x = 0$, $y = b$, and by $y = 0$, $x = a$.

Thus $\frac{x}{5} + \frac{y}{4} = 1$ is (iii) in the figure and $\frac{x}{-2} + \frac{y}{-4} = 1$ is (ii).

The equation of OX is $y = 0$; that of OY is $x = 0$.

Condition of Parallelism.

The direction of the line $Ax + By + C = 0$ is given by

$$\theta = \tan^{-1}\left(-\frac{A}{B}\right).$$

$Ax + By + C = 0$ and $A_1x + B_1y + C_1 = 0$ are parallel

$$\text{if } \frac{A}{B} = \frac{A_1}{B_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(C)}$$

for then, if θ_1 is the direction of the second line,

$$\theta_1 = \tan^{-1}\left(-\frac{A_1}{B_1}\right) = \tan^{-1}\left(-\frac{A}{B}\right) = n\pi + \theta \text{ (p. 51)},$$

and all values of n give the same or opposite directions.

Condition of Perpendicularity.

If, with the same notation, $\theta_1 = \theta \pm \frac{1}{2}\pi$, the lines are perpendicular.

In this case $\tan\theta_1 = -\cot\theta$ (p. 47).

$$\therefore \; -\frac{A_1}{B_1} = -\left(-\frac{B}{A}\right), \text{ i.e. } AA_1 + BB_1 = 0. \quad . \quad . \quad . \quad \text{(D)}$$

The converse is easily proved.

EXAMPLE. $2x + 3y = 4$ is parallel to $4x + 6y = 7$ and perpendicular to $6x - 4y = 9$.

Line through a given point in a given direction.

Let $\tan^{-1} m$ be the given direction and (x_1, y_1) the given point.

$y = mx + k$ is the equation of any line in the given direction, and $y_1 = mx_1 + k$ if the given point lies on this line.

Eliminating k by subtraction,

$$y - y_1 = m(x - x_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(E)}$$

Here x, y are the co-ordinates of any point on the line, i.e. the current co-ordinates.

Line through two Given Points.

Let (x_1, y_1), (x_2, y_2) be the given points.

Let $\tan^{-1}m$ be the unknown direction of the line.

Regarding the line as through (x_1, y_1) its equation is

$$y-y_1 = m(x-x_1).$$

It passes through (x_2, y_2) if

$$y_2-y_1 = m(x_2-x_1).$$

Eliminate m from the first equation by means of the second,

$$\frac{y-y_1}{x-x_1} = m = \frac{y_2-y_1}{x_2-x_1}.$$

This is most easily remembered as

$$\frac{\mathbf{x}-\mathbf{x}_1}{\mathbf{x}_2-\mathbf{x}_1} = \frac{\mathbf{y}-\mathbf{y}_1}{\mathbf{y}_2-\mathbf{y}_1}. \quad \text{(F)}$$

Line through a given point $(\mathbf{x}_1, \mathbf{y}_1)$ parallel to, or perpendicular to, $\mathrm{A}\mathbf{x}+\mathrm{B}\mathbf{y}+\mathrm{C} = 0$.

Using equation (E), $m = -\frac{\mathrm{A}}{\mathrm{B}}$ for parallelism, $+\frac{\mathrm{B}}{\mathrm{A}}$ for perpendicularity.

The equation of a parallel line is then

$$\mathrm{A}(\mathbf{x}-\mathbf{x}_1)+\mathrm{B}(\mathbf{y}-\mathbf{y}_1) = 0,$$

and the equation of a perpendicular line is

$$\mathrm{B}(\mathbf{x}-\mathbf{x}_1)-\mathrm{A}(\mathbf{y}-\mathbf{y}_1) = 0. \quad \text{(G)}$$

The two *lines* $\mathrm{A}x+\mathrm{B}y+\mathrm{C} = 0$, $\mathrm{A}_1x+\mathrm{B}_1y+\mathrm{C}_1 = 0$ *intersect* at the point found by solving these as simultaneous equations giving x and y, viz. $x = \frac{\mathrm{BC}_1-\mathrm{CB}_1}{\mathrm{AB}_1-\mathrm{BA}_1}$, $y = \frac{\mathrm{CA}_1-\mathrm{AC}_1}{\mathrm{AB}_1-\mathrm{BA}_1}$.

EXAMPLES.

[*In every case draw the lines obtained on a diagram.*]

1. Write down the equation of the line through (6, 4) (−3, −4), and find its direction, and the intercepts on the axes.

2. Find the equations of the line through (−2, 3), (i) parallel to, (ii) perpendicular to, the line making intercepts 4 and 5 on OX and OY respectively.

3. If (1, 2) (2, −3) (3, −4) are the vertices of a triangle, find the equations of the perpendiculars from the vertices on the sides, and verify that the point of intersection of the first two lies on the third.

4. In the same triangle find the co-ordinates of the middle points of the sides, write down the equations of the lines joining each to the vertex opposite, and verify that they are concurrent.

5. Show that the line joining the middle points of the sides of a triangle is parallel to the remaining side.

Perpendicular distance of a point R (x_1, y_1) from a line PQ $(Ax + By + C = 0.)$

Let M $(x'y')$ be the foot of the perpendicular.

The equation of the perpendicular, RM, is

$$B(x-x_1) - A(y-y_1) = 0.$$

Then $x'y'$ is on PQ and PM.

$$\therefore \quad Ax' + By' = -C,$$

and $$Bx' - Ay' = Bx_1 - Ay_1.$$

$$\therefore \quad (A^2+B^2)\,x' = -AC + B^2x_1 - ABy_1$$

and $$(A^2+B^2)\,y' = -BC + A^2y_1 - ABx_1.$$

$$\therefore \quad (A^2+B^2)(x'-x_1) = -AC - A^2x_1 - ABy_1 = -A\,(Ax_1+By_1+C).$$

and $$(A^2+B^2)(y'-y_1) = -BC - B^2y_1 - ABx_1 = -B\,(Ax_1+By_1+C)$$

The length RM $= \sqrt{\{(x'-x_1)^2 + (y'-y_1)^2\}}$

$$= \sqrt{\left\{\left(\frac{Ax_1+By_1+C}{A^2+B^2}\right)^2 \left((-A)^2 + (-B)^2\right)\right\}}$$

$$= \pm\frac{Ax_1 + By_1 + C}{\sqrt{A^2+B^2}} \quad \ldots\ldots\ldots\ldots \quad \text{(H)}$$

Hence the distance of the origin O $(0, 0)$ from the line is

$$\pm\frac{C}{\sqrt{A^2+B^2}}.$$

E.g. The distance of (2, 3) from $3x - 4y = 5$ is

$$\pm\frac{3\times2 - 4\times3 - 5}{\sqrt{3^2+4^2}} = \pm\frac{-11}{5}.$$

The distance, when (as is very frequently the case) there is no convention as to whether RM is to be considered as positive or negative, is simply $\frac{11}{5}$.

The Area of a Triangle.

Let $(x_1 y_1)$, $(x_2 y_2)$, $(x_3 y_3)$ be the angular points R, Q, P.

Then in the same notation as the last paragraph,

$$\text{Area} = \tfrac{1}{2} . RM . PQ.$$

The equation of PQ is $\dfrac{x - x_2}{x_3 - x_2} = \dfrac{y - y_2}{y_3 - y_2}$.

i.e. $\quad x(y_3 - y_2) - y(x_3 - x_2) + x_3 y_2 - x_2 y_3 = 0.$

Comparing with $\text{A}x \quad + \text{B}y \qquad + \text{C} \qquad = 0$

we have $\quad RM = \pm \dfrac{x_1(y_3 - y_2) - y_1(x_3 - x_2) + x_3 y_2 - x_2 y_3}{\sqrt{(y_3 - y_2)^2 + (x_3 - x_2)^2}}$.

But the denominator $= PQ$.

$$\therefore \text{ Area} = \pm \tfrac{1}{2}\{x_1(y_3 - y_2) - y_1(x_3 - x_2) + x_3 y_2 - x_2 y_3\}$$
$$= \pm \tfrac{1}{2}\{x_2 y_1 - x_1 y_2 + x_3 y_2 - x_2 y_3 + x_1 y_3 - x_3 y_1\}.$$

The number of units of area is the positive value of this expression.

The Angle between two Lines.

Write the lines in the form $y = mx + k$, $y = m'x + k'$, where $m = \tan\theta$, $m' = \tan\theta'$.

Then $\tan(\theta' - \theta)$

$$= \frac{\tan\theta' - \tan\theta}{1 + \tan\theta . \tan\theta'} \text{ (p. 58)} = \frac{m' - m}{1 + mm'} = \tan\alpha, \text{ say,}$$

and $\theta' - \theta = n\pi + \alpha$.

This gives the angle through which the first written line must be rotated in a positive direction to become parallel to the second.

The tangent of the acute angle between the lines is the positive value of $\pm \dfrac{m' - m}{1 + mm'}$. . . . . (I)

It is easily shown that this may also be written $\pm \dfrac{\text{AB}' - \text{BA}'}{\text{AA}' + \text{BB}'}$.

Example. To find the angle between

$$3x + 4y = 5 \text{ and } 2x - 3y + 2 = 0.$$

These are $y = -\frac{3}{4}x + \frac{5}{4}$ and $y = \frac{2}{3}x + \frac{2}{3}$.

$$\tan(\theta' - \theta) = \frac{\frac{2}{3} - \left(-\frac{3}{4}\right)}{1 + \frac{2}{3}\left(-\frac{3}{4}\right)} = \tfrac{17}{6} = \tan 70\tfrac{1}{2}°.$$

The required angle is $n \cdot 180° + 70\frac{1}{2}° = 70\frac{1}{2}°$ or $-180° + 70\frac{1}{2}°$ $= -109\frac{1}{2}°$, or $250\frac{1}{2}°$, &c.

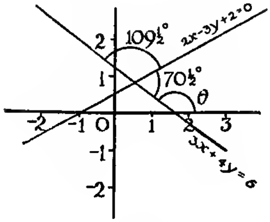


FIG. 52.

The figure explains the apparent ambiguity of the result.

No progress can be made in this subject till formulae (A) to (I) are thoroughly understood and remembered.

We give the following as an example of method.

To find the locus of a point which moves so that its perpendicular distances from two given lines are equal to each other.

Let (ξ, η) be the co-ordinates of the point in any position. Let $Ax + By + C = 0$, $A'x + B'y + C' = 0$ be the given lines.

The condition is $\pm \dfrac{A\xi + B\eta + C}{\sqrt{A^2 + B^2}} = \pm \dfrac{A'\xi + B'\eta + C'}{\sqrt{A'^2 + B'^2}}$ from (H).

$$\therefore \; (\xi, y) \text{ lies on the line } \frac{Ax + By + C}{\sqrt{A^2 + B^2}} = \frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}},$$

$$\text{or on the line } \frac{Ax + By + C}{\sqrt{A^2 + B^2}} = -\frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}},$$

and these two lines are its locus.

Conversely, every point on these lines satisfies the given condition.

From elementary geometry it follows that these lines are the bisectors of the angles formed by the first pair.

The expression $Ax' + By' + C$ is shown to be of the same sign as C, when $P(x', y')$ and the origin, O, are on the same side of the line $Ax + By + C = 0$, and of the opposite sign when P and O are on opposite sides, as follows.

If the ordinate NP meets the line in K, then, having regard to the convention of signs of lines,

$$A \cdot ON + B \cdot NK + C = 0.$$

$$\therefore \; A \cdot ON + B(NP + PK) + C = 0.$$

$$\therefore \; -B \cdot PK = Ax' + By' + C.$$

But if the line meet OY in L, $OL = -\frac{C}{B}$;

$$\therefore \frac{PK}{OL} = \frac{Ax' + By' + C}{C}.$$

Hence $Ax' + By' + C$ and C have the same sign or not, according as PK and OL have the same signs or not, i.e. according as PK and OL are drawn in the same direction or not.

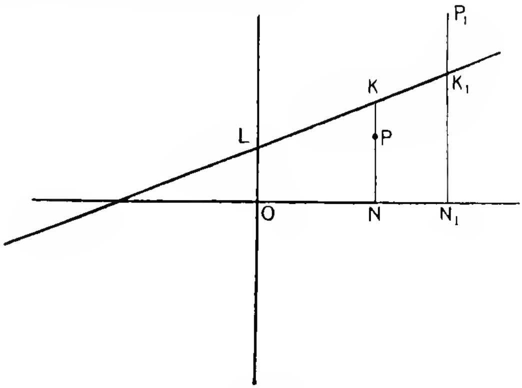


FIG. 53.

Of course if P is on the line $Ax' + By' + C$ is zero.

If now we write the perpendicular from P as $+\frac{Ax' + By' + C}{\sqrt{A^2 + B^2}}$, and take the positive root in the denominator, it is easy to deduce that all perpendiculars so written from points on the origin side of the line have the sign of C, and others the opposite sign.

EXAMPLES.

1. If the perpendicular from O on a line is of length p and makes an angle α with OX, show by methods of projection that any point (x, y) on the line satisfies the equation $x \cos\alpha + y \sin\alpha = p$, taking care that the proof applies for all values of α.

Show also by projection that the perpendicular distance of a point $(x_1\, y_1)$ from this line is $x_1 \cos\alpha + y_1 \sin\alpha - p$, and deduce formula (H).

2. Find the equations of the two lines through (3, 2) that are inclined at 30° to the line $3x - 5y = 8$. Draw the figure.

3. Explain the results obtained by using (I) to find the angles between $2x+3y = 4$ and $4x+6y = 0$, and between $3x = 5y$ and $5x+3y = 0$.

4. Write down the equations of the 6 bisectors of the angles of the triangle made by $y = 0$, $x = 0$, and $3x+4y = 5$. With the help of a figure find the co-ordinates of the inscribed and escribed circles of this triangle.

5. Find the locus of a point which moves so that the sum of its perpendicular distances from three given straight lines is constant.

Change of Origin, the Axes remaining in their original directions.

It is frequently necessary to use co-ordinates measured on two different sets of axes in the same analysis.

Let x, y be the co-ordinates of a point P measured on OX, OY.

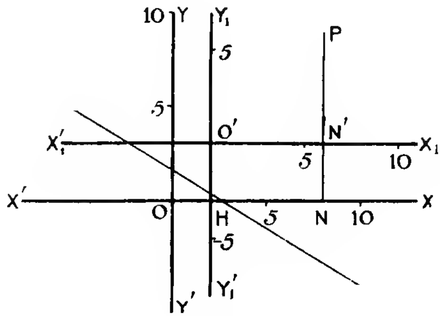


FIG. 54.

Let O' be a point h, k, and $X_1'O'X_1$, $Y_1'OY_1$ lines parallel to OX and OY, and let Y_1Y_1' meet XX' in H. Then $OH = h$, $HO' = k$.

Let NP, the ordinate of P, meet $O'X_1$ in N'.

Write x' for $O'N'$ and y' for $N'P$.

Then for all positions of P, and whatever the values of h and k,

$$x = ON = OH+HN = OH+O'N' = h+x',$$

and $$y = NP = NN'+N'P = HO'+N'P = k+y'.$$

If $f(x, y) = 0$ is any relation between x and y, that is the equation of some locus of P, then $f\{(h+x'), (k+y')\} = 0$ represents the same locus, the equation now involving x', y' instead of x, y.

While we are dealing with x', y' we can suppress the $''$ for convenience and write the equation $f\{(h+x), (k+y)\} = 0$. The equation is then said to be referred to the new axes, $O'X_1$, $O'Y_1$.

Thus if $h = 2$, $k = 3$, the line $2x+3y = 5$ referred to the old axes, becomes $2(x+2)+3(y+3) = 5$, i.e. $2x+3y+8 = 0$ referred to the new.

As in Figure 54, this line cuts the old axes at $x = 2\frac{1}{2}$, $y = 1\frac{2}{3}$, and the new at -4, $-2\frac{2}{3}$.

THE EQUATION OF THE SECOND DEGREE.

The most general form in which the equation of the 2nd degree can be written is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

where a, b, c, f, g, h are any constants.

[The apparently arbitrary order of the letters and the introduction of 2 in the coefficients do not diminish the generality, and, as will be seen in the sequel, are convenient.]

We have already used the particular cases when $b = 0$ (pp. 83–5) and when $h = b = 0$ (p. 75).

We will first take the case when $h = 0$, the other letters having any values. h is introduced again on p. 175.

The equation $\mathbf{ax^2 + by^2 + 2gx + 2fy + c = 0}$.

This may be written

$$a\left(x + \frac{g}{a}\right)^2 + b\left(y + \frac{f}{b}\right)^2 = \frac{g^2}{a^2} + \frac{f^2}{b^2} - c,$$

unless a or b is zero.

Let C be the point $\left(-\frac{g}{a}, -\frac{f}{b}\right)$.

As on p. 142 transfer to C as origin.

The equation referred to axes through C, parallel to the original axes is

$$a\left\{\left(x - \frac{g}{a}\right) + \frac{g}{a}\right\}^2 + b\left\{\left(y - \frac{f}{b}\right) + \frac{f}{b}\right\}^2 = \frac{g^2}{a^2} + \frac{f^2}{b^2} - c,$$

i.e. $$ax^2 + by^2 = \frac{g^2}{a^2} + \frac{f^2}{b^2} - c.$$

[This may be seen from Figure 55. If $OH = -\dfrac{g}{a}$, $HO = \dfrac{g}{a}$; $x + \dfrac{g}{a} = HO + x = x'$.]

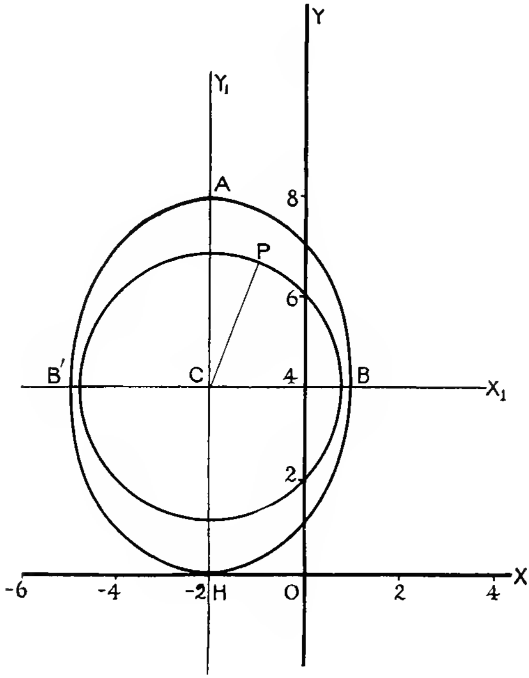


Fig. 55.

The last equation can be re-written,

$$\mathbf{A}x^2 + \mathbf{B}y^2 = 1, \text{ unless } \frac{g^2}{a} + \frac{f^2}{b} - c = 0,$$

where C is now the origin.

$$\mathbf{A} = a\Big/\left(\frac{g^2}{a} + \frac{f^2}{b} - c\right) \text{ and } \mathbf{B} = b\Big/\left(\frac{g^2}{a} + \frac{f^2}{b} - c\right).$$

The Circle.

Case I. **A** and **B** positive and equal. (Then $a = b$.)

Let $r^2 = \dfrac{1}{\mathbf{A}}$, r real but not necessarily rational.

The equation becomes $x^2+y^2=r^2$.

This is evidently $CP^2=r^2$, and the locus of P is a circle, radius r.

EXAMPLE. Let the original equation be

$$2x^2+2y^2+8x-16y+25 = 0, \text{ referred to } OX,\ OY.$$

This may be written

$$2(x+2)^2+2(y-4)^2 = 2\times 2^2+2\times 4^2-25 = 15,$$

that is, $2x^2+2y^2 = 15$ referred to CX_1, CY_1, where C is $(-2, 4)$

$$x^2+y^2 = \tfrac{15}{2} = (2{\cdot}74\ldots)^2.$$

This circle is drawn in Figure 55.

The equation then represents a circle, centre $(-2, 4)$, radius $2{\cdot}74\ldots$.

It is otherwise evident that

$$(2{\cdot}74\ldots)^2 = CP^2 = (x-(-2))^2+(y-4)^2, \qquad \text{(A) p. 131.}$$

i.e. $2x^2+2y^2+8x-16y+25 = 0$, is the equation of a circle, centre $(-2, 4)$, radius $2{\cdot}74\ldots$.

The Ellipse.

CASE II. **A** and **B** positive and unequal. The curve is then called an ellipse.

Let $\mathbf{A}=\dfrac{1}{\alpha^2}$ and $\mathbf{B}=\dfrac{1}{\beta^2}$, where α and β are real, but not necessarily rational.

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1.$$

$|y| \ngtr \beta$, or x^2 would be negative. $|x| \ngtr \alpha$, or y^2 would be negative.

Mark the points $A(\alpha, 0)$, $B(0, \beta)$, $A'(-\alpha, 0)$, $B'(0, -\beta)$ Fig. 56.

Then the curve is entirely within the rectangle formed by lines through A and A' parallel to CY, and through B and B' parallel to CX.

The longer of the two lines AA', BB' is the *transverse* or *major axis*, the shorter the *minor axis*. A, A', B, B' are the *vertices*, C the *centre*, and CA, CB the *semi-axes*.

If any point P $(x_1\, y_1)$ satisfies the equation, then P_1 $(-x_1\, y_1)$,

$P_2(-x_1 \; -y_1)$, and $P_3(x_1 \; -y_1)$ also satisfy it, since only the squares of the co-ordinates are involved. The curve is then symmetrical with regard to both CX and CY.

If $x_1 = CM$, $y_1 = MP$, we have

$$\frac{y_1^{\,2}}{\beta^2} = 1 - \frac{x_1^{\,2}}{\alpha^2} = \frac{(\alpha - x_1)(\alpha + x_1)}{\alpha^2}.$$

Following the convention of signs of lines,

$$\frac{MP^2}{CB^2} = \frac{(CA - CM)(A'C + CM)}{CA^2} = \frac{A'M \cdot MA}{CA^2}.$$

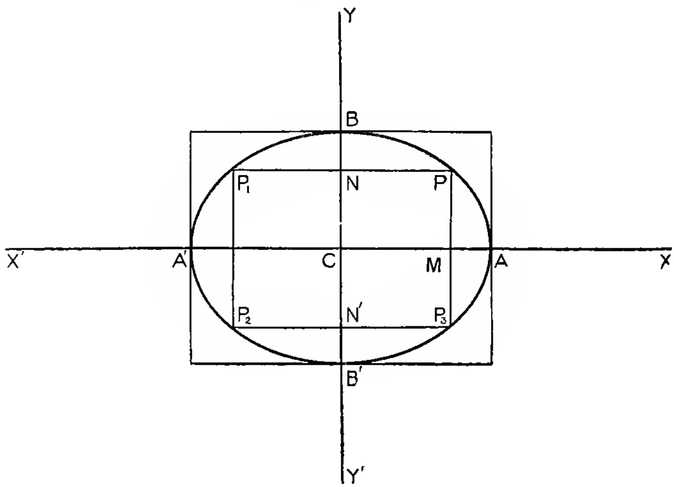


FIG. 56.

[In the circle the corresponding property is $MP^2 = A'M \cdot MA$.]

EXAMPLE. $16x^2 + 9y^2 + 64x - 72y + 64 = 0$,

$$16(x+2)^2 + 9(y-4)^2 = 144$$

$$\left(\frac{x+2}{3}\right)^2 + \left(\frac{y-4}{4}\right)^2 = 1.$$

Centre $(-2, 4)$; semi-major axis 4; semi-minor axis 3.

The curve is shown in Figure 55. The vertices on the minor axis are called B, B', as is usually done.

The Hyperbola.

CASE III. **A** and **B** of different signs. The curve is then called an hyperbola. [If **A** and **B** are both negative there are no real values of (x, y).]

Let $\mathbf{A} = \frac{1}{\alpha^2}$, $\mathbf{B} = -\frac{1}{\beta^2}$, where α and β are real.

$$\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1.$$

$|x| \nless \alpha$, or y^2 would be negative. The curve is entirely outside the lines through A, A' parallel to CY, where $A'C = CA = \alpha$.

y can have any value from $-\infty$ to $+\infty$.

The curve is symmetrical with regard to CX and CY.

Let x_1 (CM) be any positive value of x greater than α.

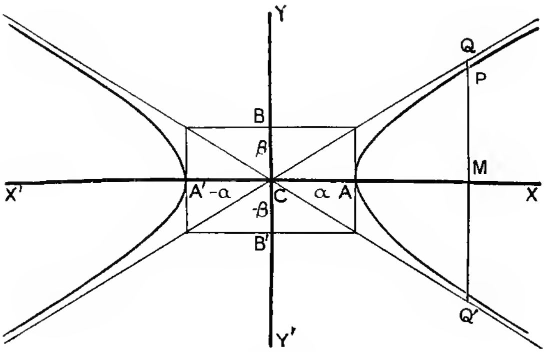


FIG. 57.

Draw the line $\frac{x}{\alpha} = \frac{y}{\beta}$ (CQ), and let Q be $(x_1\ y_2)$, where $y_2 = \frac{\beta}{\alpha} x_1 = MQ$ and y_2 is positive.

Let $(x_1 y_1)$ P be on the curve, y_1 being positive and $= MP$.

Then $MQ^2 - MP^2 = {y_2}^2 - {y_1}^2 = \left(\frac{\beta}{\alpha} x_1\right)^2 - \beta^2\left(\frac{{x_1}^2}{a^2} - 1\right) = \beta^2$

$$PQ = MQ - MP = \frac{\beta^2}{MQ + MP}.$$

As CM and $\therefore$ MP and MQ increase, PQ diminishes indefinitely, Q remaining above P. The curve therefore approaches the line CQ, becomes indefinitely near it, but never crosses it.

From the symmetry of the curve a similar property appears in each of the four quadrants, as in the figure, twice with reference to the line $\frac{x}{\alpha} = \frac{y}{\beta}$ and twice with $\frac{x}{\alpha} = -\frac{y}{\beta}$. These lines $\frac{x}{\alpha} = \pm\frac{y}{\beta}$ are called the *asymptotes* (see p. 147) of the curve. $A'A$ is its *transverse* axis, A, A' its vertices, C its centre. If $\alpha = \beta$, the asymptotes are at right angles, and the curve is called a *rectangular* hyperbola.

Now $$\frac{y^2}{\beta^2} = \frac{x^2}{\alpha^2} - 1 = \frac{(x-\alpha)(x+\alpha)}{\alpha^2}.$$

$$\therefore \quad \frac{MP^2}{CB^2} = \frac{(CM-CA)(A'C+CM)}{CA^2} = \frac{AM \,.\, A'M}{CA^2};$$

where CB, CB' are cut off CY, $CY' = \beta$.

Lines through B, B' parallel to $X'X$ evidently meet the lines through AA' parallel to $Y'Y$ on the asymptotes.

It is easy to show that $-\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$ is a hyperbola with transverse axis $B'B$, and the same asymptotes as $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$.

These curves are said to be conjugate to each other.

EXAMPLE. Find the centre of the curve

$$9x^2 - 16y^2 - 36x - 128y = 364,$$

and trace it, and show that its conjugate is

$$9x^2 - 16y^2 - 36x - 128y = 76.$$

(Write the equation referred to the centre and then transfer back to the first origin.)

The Parabola.

Returning to the equation $ax^2 + by^2 + 2gx + 2fy + c = 0$, take the case where $a = 0$.

Then $$b\left(y + \frac{f}{b}\right)^2 = -2gx - c + \frac{f^2}{b},$$

$$\left(y + \frac{f}{b}\right)^2 = -\frac{2g}{b}\left\{x + \frac{c}{2g} - \frac{f^2}{2gb}\right\}.$$

Refer to parallel axes through a point A whose co-ordinates are

$$\left(-\frac{c}{2g}+\frac{f^2}{2gb}\right) \text{ and } \left(-\frac{f}{b}\right),$$

and the equation becomes

$$y^2 = -\frac{2g}{b}x = 4px, \text{ where } p = -\frac{g}{2b}.$$

This curve is a *parabola*; A is its vertex, and AX_1 its axis.

EXAMPLE. $3y^2 - 13x - 12y - 27 = 0$,
$3(y-2)^2 = 13x + 27 + 12$,
$(y-2)^2 = \frac{13}{3}(x+3)$.

Referred to $(-3, 2)$, $y^2 = \frac{13}{3}x$, where AX_1, AY_1 are the axes of co-ordinates. x must always be positive, and the curve lies completely to the right of AY_1. (To the left if g and b had been of the same sign.)

It is symmetrical with regard to AX_1.

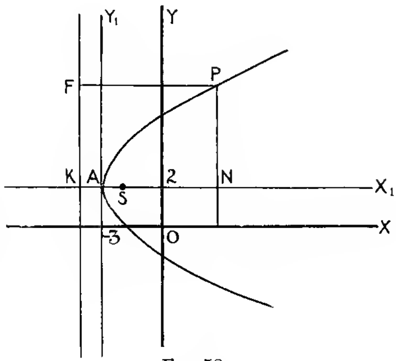


FIG. 58.

Cut off $AS = p = \frac{13}{12}$.

$NP^2 = y^2 = 4AS \cdot x = 4AS \cdot AN$, if NP is the ordinate of a point on the curve.

AX_1 is called the *axis* of the parabola.

If b were zero instead of a, a similar analysis would apply, but the axis would be parallel to the axis OY instead of OX.

The equation is $y = -\frac{a}{2f}x^2 - \frac{g}{f}x - \frac{c}{2f}$, that is, it represents the quadratic function already discussed on pp. 75-9.

If f and a are of different signs the U-shape is upwards, if of the same sign, downwards.

EXAMPLES. 1. Draw on the same diagram $y^2 = 16x$, $y^2 = -16x$, $x^2 = 16y$, $x^2 = -16y$.

2. Draw the curves (i) $4y^2 + 6x - 8y = 0$,

(ii) $4x^2 - 8x + 6y = 0$.

If in the equation on p. 143 $\frac{g^2}{a}+\frac{f^2}{b}-c=0$, the curve is $ax^2+by^2=0$.

If a and b are of the same sign no points (except 0, 0) can be found.

If a and b are of opposite sign, we have

$$y = \pm\sqrt{-\frac{a}{b}}\cdot x,$$

a pair of straight lines, asymptotes of the hyperbola

$$ax^2-(-b)y^2=k,$$

where k is any constant.

We have used the terms ellipse, hyperbola, and parabola without exact definitions. These are given in the next paragraph.

Examples.

Find the centre, vertices, asymptotes, and the equations and lengths of axes of the following curves and draw them:

1. $x^2+y^2+4x=0$.
2. $9x^2+25y^2=225$.
3. $9x^2-25y^2=225$.
4. $25x^2-9y^2=225$.
5. $4x^2+y^2+8x-10y=0$.
6. $8x^2-16y^2+15x+30=0$.
7. $4x^2+8y^2+12y=17$.

8. Show that all the curves $\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=k$ have the same asymptotes, when α and β are fixed and k is variable, and that they are conjugate in pairs for equal and opposite values of k.

9. Show that the expression $\mathbf{A}x'^2+\mathbf{B}y'^2-1$ is positive or negative according as the $x'\,y'$ is within or without the curve $\mathbf{A}x^2+\mathbf{B}y^2=1$. Explain the result when $\mathbf{A}$ and $\mathbf{B}$ are of opposite signs. Obtain a similar theorem for the parabola.

Focus and Directrix.

Definition. If a point P moves so that its distance from a fixed point bears a constant ratio to its distance from a fixed line, its locus is called an ellipse, parabola, or hyperbola, according as this ratio is less than, equal to, or greater than unity.

The fixed point (S) is called the *focus*, the fixed line (KF) the *directrix*, the ratio (e) the *eccentricity*.

If PF be the perpendicular on to the directrix, $SP = ePF$.

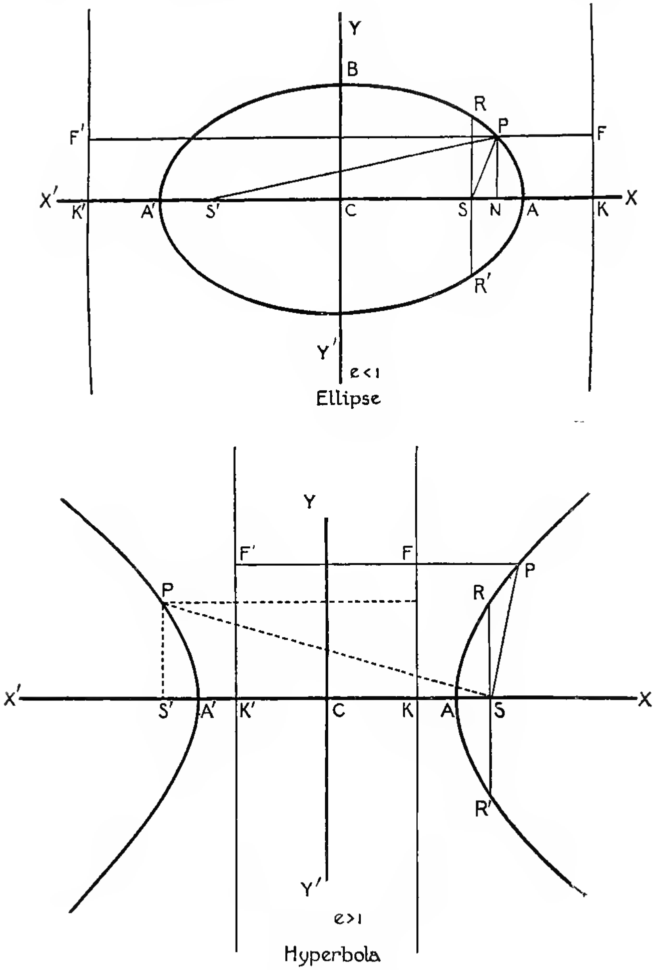


FIG. 59.

We shall now show that the curves obtained from the equation $ax^2 + by^2 + 2gx + 2fy + c = 0$ all satisfy this definition and that the names used in the last section are in accordance with it.

I. When neither a, b nor $\frac{g^2}{a}+\frac{f^2}{b}-c$ is zero, we obtain the equation $\mathbf{A}x^2+\mathbf{B}y^2=1$.

If $\mathbf{A}$ and $\mathbf{B}$ are both negative no real points are obtained.

Let $\mathbf{A}$ be positive,* and equal to $\frac{1}{\alpha^2}$; and if $\mathbf{B}$ is also positive let $\mathbf{A}<\mathbf{B}$.*

Let $e=$ the positive value of $\sqrt{1-\frac{\mathbf{A}}{\mathbf{B}}}$, so that

$$e^2=1-\frac{\mathbf{A}}{\mathbf{B}},\quad \frac{\mathbf{A}}{\mathbf{B}}=1-e^2.$$

$$\mathbf{A}x^2+\mathbf{B}y^2=1=\mathbf{A}\alpha^2.$$

Divide by $\mathbf{B}$, $\quad (1-e^2)(x^2-\alpha^2)+y^2=0,$

$$(x\mp\alpha e)^2+y^2=e^2\left(x\mp\frac{\alpha}{e}\right)^2.$$

Let S, S' be the points $(0, \pm\alpha e)$, and KF, $K'F'$ the lines $x=\pm\frac{\alpha}{e}$, where K, K' are on XX'.

Let P be any point on the locus and PF, PF' perpendiculars on to KF, $K'F'$.

Then the equation just given is, when the upper signs are taken, $SP^2=e^2PF^2$, $SP=ePF$, and when the lower signs are taken, $S'P^2=e^2PF'^2$, $S'P=ePF'$, SP and PF being taken as positive lengths and e being positive by definition.

The curve may then be traced either with S, KF or with S', $K'F'$ as focus and directrix.

If $\mathbf{B}$ is positive, and $=\frac{1}{\beta^2}$, we have the curve $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$ of p. 145, and $e^2=1-\frac{\beta^2}{\alpha^2}$, $e<1$. The curve is an ellipse.

If $\mathbf{B}$ is negative, and $=-\frac{1}{\beta^2}$, we have the curve $\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=1$ of p. 147, and $e^2=1+\frac{\beta^2}{\alpha^2}$, $e>1$. The curve is an hyperbola.

* If $\mathbf{A}$ is negative and $\mathbf{B}$ positive, or if $\mathbf{A}$ and $\mathbf{B}$ are both positive and $\mathbf{A}>\mathbf{B}$, the same analysis would be correct with $e=\sqrt{1-\frac{\mathbf{B}}{\mathbf{A}}}$ and with the foci on the axis of y. If $\mathbf{A}=\mathbf{B}$ we have a circle and $e=0$.

If the line through S parallel to KF meets the curve at R, R', RR' is called the latus rectum.

$SR = eSK = e\left(\frac{\alpha}{e} - \alpha e\right)$ in ellipse, $= e\left(\alpha e - \frac{\alpha}{e}\right)$ in hyperbola, $= \frac{\beta^2}{\alpha}$ in each case.

In the ellipse $SP = ePF = e\,(CK - CN)$, where NP is the ordinate of P, $\qquad = e\left(\frac{\alpha}{e} - x\right) = \alpha - ex,$

and $\qquad S'P = ePF' = e\left(\frac{\alpha}{e} + x\right) = \alpha + ex.$

$\therefore$ $SP + S'P = 2\alpha = AA'$ and is constant, where A, A' are the points $(\pm\alpha,\ 0)$.

In the hyperbola $SP = e\left(x - \frac{\alpha}{e}\right)$ and $S'P = e\left(x + \frac{\alpha}{e}\right)$,

and $\qquad S'P - SP = 2\alpha = AA'$ and is constant.

It is easily shown that $AS\,.\,SA' = \beta^2$ in both curves, and that $SB = \alpha$ in the ellipse, where B is $(0,\ \beta)$.

II. When a (or b) is 0, the equation

$$y^2 = 4px \text{ (or } x^2 = 4py)$$

is obtained (p. 149).

Let S be $(0, p)$, and KF be $x = -p$.

Then $SP^2 = (x-p)^2 + y^2$
$= (x-p)^2 + 4px = (x+p)^2$
$= PF^2.$

$$SP = PF.$$

The curve is therefore a parabola. $(e = 1.)$

A is the point $(0,\ 0)$.

If $x = p$, $y = \pm 2p$, and the latus rectum, RR', is $4p$.

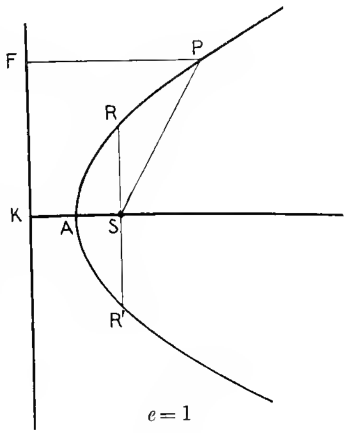


FIG. 60.

III. If $\frac{g^2}{a} + \frac{f^2}{b} - c = 0$, S and K coincide, and the locus becomes the lines $y = \pm\sqrt{\frac{-\mathbf{A}}{\mathbf{B}}}\cdot x$, the limiting form of an hyperbola when its major axis tends to zero while e remains constant.

The following geometrical investigation shows that all these curves can be obtained as plane sections of a cone. They are therefore called conic sections.

Let $A'VA_1D_1'$ be a symmetrical section through the vertex V of

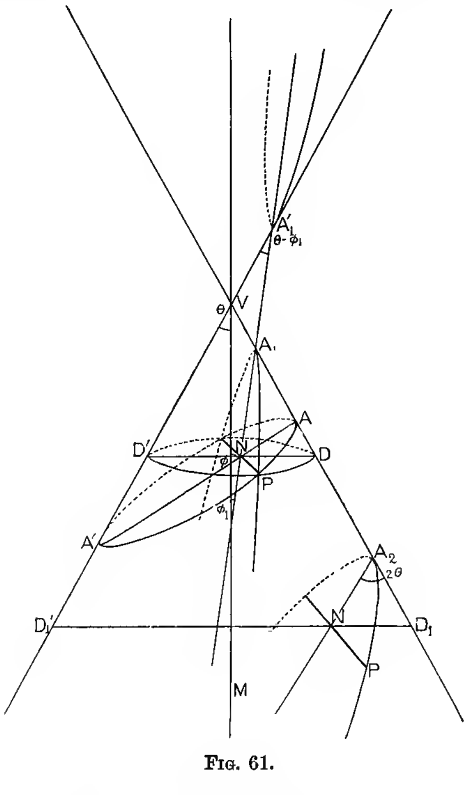


Fig. 61.

a right circular double cone. Let VM be its axis, θ its semi-vertical angle.

Let AA' be the line of symmetry of any plane section which cuts VD_1', VD_1 on the same side of V.

Let P be any point on this section, and PN perpendicular to AA'.

Elementary solid geometry shows that PN is perpendicular to the plane $D_1'VD_1$ and that the section through NP perpendicular to VM

is symmetrical and circular. Let the last-named section cut $D_1'VD_1$ at D and D'. DD' passes through N, and is a diameter of DPD'; and $PN^2 = DN \, . \, ND'$.

Let ϕ be the angle between AA' and VM.

$$\text{Then } \frac{PN^2}{AN \, . \, NA'} = \frac{DN}{AN} \cdot \frac{D'N}{A'N} = \frac{\sin(\theta+\phi)}{\cos\theta} \cdot \frac{\sin(\phi-\theta)}{\cos\theta}$$

$$= \frac{\cos^2\theta - \cos^2\phi}{\cos^2\theta} \text{(p. 60)} = 1 - \frac{\cos^2\phi}{\cos^2\theta} = 1 - e^2,$$

where $e < 1$ since $\phi > \theta$.

This is a property characterizing the ellipse (p. 150).

Let A_1A_1' be the line of symmetry of a section cutting $D_1'V$ produced and VD_1, and let ϕ_1 be the angle between $A_1'A_1$ and VM.

$$\text{Then, as before, } \frac{PN^2}{A_1N \, . \, NA_1'} = \frac{DN}{A_1N} \cdot \frac{D'N}{A_1'N} = \frac{\sin(\theta+\phi_1) \, . \, \sin(\theta-\phi_1)}{\cos^2\theta}$$

$$= \frac{\cos^2\phi_1}{\cos^2\theta} - 1 = e^2 - 1, \text{ where } e > 1, \text{ since } \theta > \phi.$$

This is a property characterizing the hyperbola.

Let the line of symmetry of the section be A_2N parallel to VA'.

$$\frac{PN^2}{A_2N} = \frac{D_1N}{A_2N} \cdot ND_1'.$$

But $\dfrac{D_1N}{A_2N}$ and $D_1'N$ are constant throughout the section.

$\therefore$ $PN^2 \propto A_2N$ and the curve is a parabola (pp. 149 and 153).

All eccentricities can be obtained from 0 up to sec θ (when $\phi_1 = 0$). Parallel sections are similar and all sizes can be obtained. If the section is through V, we have a pair of straight lines, i.e. an hyperbola with vanishing major axis.

From pp. 150–3 and 154–5 it is seen that the parabola is intermediate between the ellipse and hyperbola. It can be obtained as the limiting form of either as follows:

The ellipse referred to a vertex is

$$\frac{(x-\alpha)^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1.$$

that is, $$y^2 = 4px\left(1 - \frac{x}{2\alpha}\right), \text{ where } 2p = \frac{\beta^2}{\alpha}$$
(the semi-latus rectum).

Now increase the major axis (2α) indefinitely, keeping the vertex and the length of the latus rectum unchanged. The equation tends towards and ultimately becomes $y^2 = 4px$.

$e^2 = 1 - \frac{\beta^2}{\alpha^2} = 1 - \frac{2p}{\alpha}$ and tends to 1.

If we start with the hyperbola $\frac{(x+\alpha)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$, we obtain the same result.

INTERSECTIONS OF THE EQUATIONS OF THE FIRST AND SECOND DEGREE.

We will take the forms $y = mx + k$ for the line, $\mathbf{A}x^2 + \mathbf{B}y^2 = 1$ for a central conic, distinguishing $\frac{x^2}{\alpha^2} \pm \frac{y^2}{\beta^2} = 1$ for the ellipse and hyperbola, and $y^2 = 4px$ for the parabola.

$y = mx + k$ meets $\mathbf{A}x^2 + \mathbf{B}y^2 = 1$ at points whose abscissae are given by the quadratic equation $\mathbf{A}x^2 + \mathbf{B}(mx+k)^2 = 1$, obtained by substituting for y.

$$(\mathbf{A} + \mathbf{B}m^2)x^2 + 2\mathbf{B}mkx + \mathbf{B}k^2 - 1 = 0. \quad . \quad . \quad . \quad \text{(i)}$$

The roots are real if $(2\mathbf{B}mk)^2 - 4(\mathbf{A} + \mathbf{B}m^2)(\mathbf{B}k^2 - 1) \nless 0$,

if $\mathbf{A} + \mathbf{B}m^2 - \mathbf{AB}k^2 \nless 0$. . . . . (ii)

The roots are equal if $\mathbf{A} + \mathbf{B}m^2 = \mathbf{AB}k^2$,

$$\text{if } k = \pm\sqrt{\frac{m^2}{\mathbf{A}} + \frac{1}{\mathbf{B}}}\,.$$

The line in this case is said to touch, or to be a tangent to, the curve. If we suppose m to remain constant and k to change, then as k approaches this value (from one side or the other) there are two points of intersection, which coalesce when k reaches this value.

The tangents in the direction $\tan^{-1} m$ to the ellipse are

$$\mathbf{y} = \mathbf{mx} \pm \sqrt{\alpha^2\mathbf{m}^2 + \beta^2}\,;$$

and there are two real tangents in every direction.

The tangents to the hyperbola are

$$\mathbf{y} = \mathbf{mx} \pm \sqrt{\alpha^2\mathbf{m}^2 - \beta^2}\,;$$

these are only real if $\alpha^2 m^2 \nless \beta^2$; if $|m| \nless \frac{\beta}{\alpha}$. If $|m| = \frac{\beta}{\alpha}$ we have the asymptotes.

Similarly, the tangents to $\frac{y^2}{\beta^2} - \frac{x^2}{\alpha^2} = 1$ are only real if $\alpha^2 m^2 \ngtr \beta^2$.

Going back to equation (ii), the roots are real and unequal if

$$\mathbf{AB}k^2 < \mathbf{A} + \mathbf{B}m^2.$$

In the ellipse this becomes $|k| \ngtr \sqrt{\alpha^2 m^2 + \beta^2}$.

In the hyperbola this becomes $-\frac{k^2}{\alpha^2 \beta^2} < \frac{1}{\alpha^2} - \frac{m^2}{\beta^2}$;

$$k^2 \nless \alpha^2 m^2 - \beta^2.$$

If $|m| < \frac{\beta}{\alpha}$, this condition is always satisfied.

Referring to the figure on p. 147, we see that all lines parallel to lines through C within the angle QCQ' cut the curve in two real points; and that lines in other directions cut, touch, or do not meet the curve, according to their position as determined by the value of k.

In equation (i) if $m = \pm \sqrt{\frac{\mathbf{A}}{-\mathbf{B}}} = \pm \frac{\beta}{\alpha}$ in the hyperbola, the coefficient of x^2 is zero. This corresponds to one infinite root. [In the quadratic $ax^2 + bx + c = 0$, or $c\left(\frac{1}{x}\right)^2 + b\left(\frac{1}{x}\right) + a = 0$, one root in $\frac{1}{x}$ is zero, and therefore one root in x is ∞ if $a = 0$.]

All lines parallel to an asymptote meet the curve at infinity and at a finite point $x = \frac{1 - \mathbf{B}k^2}{2\mathbf{B}mk}$. If k is 0, the second root is also infinite, and also the condition of equality of roots is satisfied. For this reason an asymptote is said to *touch* the curve at infinity.

$y = mx + k$ meets the parabola $y^2 = 4px$ at points whose abscissae are roots of $(mx + k)^2 = 4px$,

$$m^2 x^2 + 2x(mk - 2p) + k^2 = 0.$$

The roots are equal if $(mk - 2p)^2 = k^2 m^2$,

$$mk = p.$$

$\therefore\ \mathbf{y = mx + \frac{p}{m}}$ is the tangent in direction $\tan^{-1} m$.

There is one real tangent in every direction. By putting $k = \frac{p}{m}$ we find that the point of contact is $\left(\frac{p}{m^2}, \frac{2p}{m}\right)$. If this is (x_1, y_1), the equation to the tangent may be written

$$\frac{2p}{m} y = 2px + \frac{2p^2}{m^2}, \text{ i.e. } \mathbf{yy_1 = 2p\,(x + x_1)}.$$

The roots are real and unequal

$$\text{if} \quad (mk - 2p)^2 > k^2 m^2,$$

$$\text{if} \quad k < \frac{p}{m},$$

and imaginary if $\quad k > \frac{p}{m}$.

Note that it is easy to show that the tangent at the vertex is parallel to the axis of y, as drawn in Figures 59 and 60.

$y = mx + k$ touches the curve $\mathbf{A}x^2 + \mathbf{B}y^2 = 1$ at $x_1\, y_1\, (P)$, where $x_1 = -\frac{\mathbf{B}mk}{\mathbf{A} + \mathbf{B}m^2}$, since the roots of (i) are equal, and where $y_1 = mx_1 + k = \frac{\mathbf{A}k}{\mathbf{A} + \mathbf{B}m^2}$, where $k = \sqrt{\frac{\mathbf{B}m^2 + \mathbf{A}}{\mathbf{AB}}}$.

$$\therefore\ \mathbf{A}x_1 = -\frac{\mathbf{AB}mk}{k^2\mathbf{AB}} = -\frac{m}{k} \text{ and } \mathbf{B}y_1 = \frac{\mathbf{AB}k}{\mathbf{AB}k^2} = \frac{1}{k}.$$

The equation is $-\frac{m}{k} x + \frac{y}{k} = 1$;

and $\therefore\ \mathbf{Axx_1 + Byy_1 = 1}$ is the tangent at (x_1, y_1).

P is on the line $\mathbf{A}x = -m\mathbf{B}y$, and if the equation of CP is written $y = m'x$, then $mm' = -\frac{\mathbf{A}}{\mathbf{B}}$.

The length of the semi-diameter CD in direction m is $x'^2 + y'^2$, where x', y' lies on the curve and on $y = mx$.

$$\therefore\ (\mathbf{A} + \mathbf{B}m^2)\, x'^2 = 1,$$

and $$CD^2 = \frac{1 + m^2}{\mathbf{A} + \mathbf{B}m^2}$$

The equation of a chord P, Q is obtained as follows:

If P, Q are $(x_1 y_1)$, $(x_2 y_2)$, then

$$\mathbf{A}x_1^2+\mathbf{B}y_1^2 = 1 = \mathbf{A}x_2^2+\mathbf{B}y_2^2;$$

$$\therefore\ \mathbf{A}(x_1^2-x_2^2) = -\mathbf{B}(y_1^2-y_2^2).$$

Equation of PQ is $\dfrac{x-x_2}{x_1-x_2} = \dfrac{y-y_2}{y_1-y_2}$.

Combining these equations,

$$\mathbf{A}(x-x_2)(x_1+x_2) = -\mathbf{B}(y-y_2)(y_1+y_2)$$

unless $x_1 = x_2$.

$$\therefore\ \mathbf{A}\mathbf{x}(\mathbf{x}_1+\mathbf{x}_2)+\mathbf{B}\mathbf{y}(\mathbf{y}_1+\mathbf{y}_2) = \mathbf{A}x_2^2+\mathbf{B}y_2^2+\mathbf{A}x_1x_2+\mathbf{B}y_1y_2$$
$$= 1+\mathbf{A}\mathbf{x}_1\mathbf{x}_2+\mathbf{B}\mathbf{y}_1\mathbf{y}_2.$$

This is the equation of the chord in a symmetrical form.

We can obtain the tangent at P as follows:

Let $x_2 = x_1+d$, $y_2 = y_1+d'$,

$$2\mathbf{A}xx_1+2\mathbf{B}yy_1+\mathbf{A}d(x-x_1)+\mathbf{B}d'(y-y_1) = 1+\mathbf{A}x_1^2+\mathbf{B}y_1^2 = 2.$$

Now let d and therefore d' become small and proceed to the limit where Q is indistinguishable from P. We obtain

$$\mathbf{A}xx_1+\mathbf{B}yy_1 = 1,$$

as before, for the equation of the tangent.

Conjugate Diameters.

If $Q_1(x_1 y_1)$ and $Q_2(x_2 y_2)$ are the intersections of $y = mx+k$ and $\mathbf{A}x^2+\mathbf{B}y^2 = 1$, and $V(\bar{x}, \bar{y})$ is the middle point of Q_1, Q_2,

then $\bar{x} = \dfrac{x_1+x_2}{2}$ = half sum of roots of equation (i), p. 156,

$$= -\frac{\mathbf{B}mk}{\mathbf{A}+\mathbf{B}m^2}.$$

But V is on $Q_1 Q_2$; $\therefore\ \bar{y} = m\bar{x}+k$.

Now regard m as constant and k as variable; that is, take a series of parallel lines to meet the curve.

Eliminating k, $\bar{x}(\mathbf{A}+\mathbf{B}m^2) = -\mathbf{B}m(\bar{y}-m\bar{x})$,

and V is on the line $\mathbf{A}x+\mathbf{B}my = 0$, which may be written

$$y = m'x, \text{ where } \mathbf{m}\mathbf{m}' = -\frac{\mathbf{A}}{\mathbf{B}}.$$

Hence (p. 158) m' is the direction of the tangent at P, where CP is in the direction m.

The locus of the middle points of chords parallel to a diameter

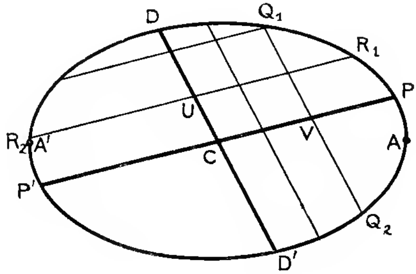


Fig. 62.

PP' is therefore the diameter (DD') parallel to the tangent at P.*

The symmetry of the equation $mm' = -\frac{\mathbf{A}}{\mathbf{B}}$, and the wording of the proposition, both show that the property is mutual; i.e. that while DD' bisects chords parallel to PP', PP' bisects chords parallel to DD'.

In the ellipse, lines through C in both directions m and m' meet the curve. DD' and PP' are then called *conjugate diameters.* $\mathbf{mm}' = -\frac{\beta^2}{\alpha^2}$.

In the hyperbola, $mm' = -\frac{\mathbf{A}}{\mathbf{B}} = \frac{\beta^2}{\alpha^2}$. If $m < \frac{\beta}{\alpha}$, CP meets the curve (p. 157); hence $m' > \frac{\beta}{\alpha}$ and CD does not meet it. CP, CD are then *conjugate directions.*

If the hyperbola degenerates into a pair of straight lines, the properties relating to the bisection of parallel chords and conjugate directions is still true.

In the circle $mm' = -1$, and perpendicular diameters are conjugate.

* It is also clear that the tangent at P is the limiting position of a parallel chord.

In the parabola it is easily shown that the locus of middle points of chords in direction m is $y = \frac{2p}{m}$, that is, the line parallel to the axis through the point of contact of the parallel tangent

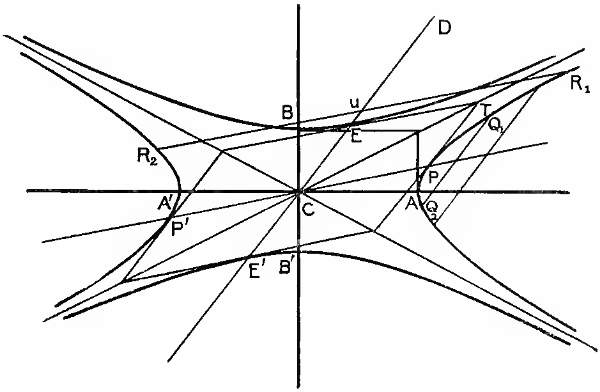


FIG. 63.

$y = mx + \frac{p}{m}$. Such lines are called diameters of the parabola.

[The property can also be obtained by keeping R fixed in Figures 59 and letting C go to infinity in the direction A'.]

In the ellipse $CP^2 + CD^2 = \frac{1+m^2}{\mathbf{A}+\mathbf{B}m^2} + \frac{1+m'^2}{\mathbf{A}+\mathbf{B}m'^2}$ (p. 158)

$$= \frac{1+m^2}{\mathbf{B}(-m'+m)\,m} + \frac{1+m'^2}{\mathbf{B}(-m+m')\,m'} = \frac{m'(1+m^2)-m(1+m'^2)}{\mathbf{B}mm'(m-m')}$$

$$= \frac{mm'-1}{\mathbf{B}mm'} = \frac{1}{\mathbf{B}} + \frac{1}{\mathbf{A}} = \alpha^2+\beta^2, \text{ and is constant.}$$

Application to the Conjugate Hyperbola, viz.

$$-\mathbf{A}x^2 - \mathbf{B}y^2 = 1. \quad \mathbf{A} = \frac{1}{\alpha^2}, \ \mathbf{B} = -\frac{1}{\beta^2}.$$

This hyperbola has β for its semi-axis, B for its vertex, and the same asymptotes, $y = \pm\frac{\beta}{\alpha}x$, as the hyperbola $\mathbf{A}x^2 + \mathbf{B}y^2 = 1$.

If $mm' = -\frac{\mathbf{A}}{\mathbf{B}} = -\frac{-\mathbf{A}}{-\mathbf{B}}$, the directions m, m' are conjugate for both hyperbolas.

Let the line $y = m'x$ meet the conjugate hyperbola in E; we have from p. 158 $CE^2 = \frac{1+m'^2}{-\mathbf{A}-\mathbf{B}m'^2}$, and working as in the last paragraph, $CP^2 - CE^2 = \frac{1}{\mathbf{A}} + \frac{1}{\mathbf{B}} = \alpha^2 - \beta^2$, and is constant.

From p. 158 the tangent at P, on the diameter $y = mx$, is in direction m', and its equation is

$$y = m'x + \sqrt{\alpha^2 m'^2 - \beta^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Similarly the tangent at E, on the diameter $y = m'x$ of the conjugate hyperbola, is in the direction given by $mm' = -\frac{-\mathbf{A}}{-\mathbf{B}}$, and is parallel to CP. Its equation is

$$y = mx + \sqrt{-\frac{m^2}{\mathbf{A}} - \frac{1}{\mathbf{B}}} = mx + \sqrt{-\alpha^2 m^2 + \beta^2}. \quad . \quad . \quad \text{(ii)}$$

(i) and (ii) can be shown to meet on the asymptotes $y = \frac{\beta}{\alpha}x$. (iii)

For the abscissae of the intersections given by (i) and (iii) are

$$x^2\left(\frac{\beta}{\alpha} - m'\right)^2 = \alpha^2 m'^2 - \beta^2, \quad x^2 = \alpha^2 \frac{\alpha m' + \beta}{-\beta + \alpha m'} = \alpha^2 \frac{\beta^2 + \alpha\beta m}{-\alpha\beta m + \beta^2};$$

by (ii) and (iii) are $x^2\left(\frac{\beta}{\alpha} - m\right)^2 = -\alpha^2 m^2 + \beta^2$, $\quad x^2 = \alpha^2 \cdot \frac{\beta + \alpha m}{\beta - \alpha m}$.

If T is the point of intersection, $PT = CE$, and $ET = CP$.

Thus the asymptotes are the diagonals of the parallelogram formed by tangents to the two hyperbolas in conjugate directions.

Examples. 1. Show that all hyperbolas having the same asymptotes have the same conjugate directions.

2. If a line meets an hyperbola, its asymptotes, and its conjugate in $Q_1 Q_2$, $L_1 L_2$, and $K_1 K_2$ respectively, then $L_1 Q_1 = L_2 Q_2$ and $L_1 K_1 = L_2 K_2$.

Pole and Polar.

Let the tangents at $K\ (x'\,y')$, $K'\ (x''\,y'')$ to $Ax^2 + By^2 = 1$ meet at $Q\ (\xi, \eta)$, and let KK' always pass through the fixed point $P\ (x_1, y_1)$. We shall prove that the locus of Q, as the chord KK' takes various positions, is $\mathbf{A}xx_1 + \mathbf{B}yy_1 = 1$. This line is called the polar of P with reference to the curve, and P is the pole of the locus.

The equations of the tangents KQ, $K'Q$ are

$$\mathbf{A}xx' + \mathbf{B}yy' = 1,$$
$$\mathbf{A}xx'' + \mathbf{B}yy'' = 1.$$

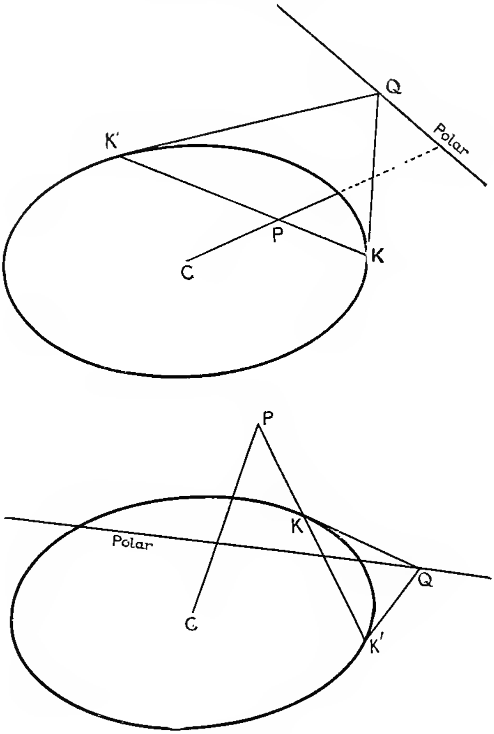


FIG. 64.

Since these pass through Q,

$$\mathbf{A}\,\xi x' + \mathbf{B}\eta y' = 1 \quad \text{and} \quad \mathbf{A}\,\xi x'' + \mathbf{B}\eta y'' = 1.$$

But these are the conditions that x', y'; x'', y'' lie on $\mathbf{A}\,\xi x + \mathbf{B}\eta y = 1$, which is therefore the equation of KK', the chord of contact of the tangents from Q.

Since P is on KK', $\quad \mathbf{A}\,\xi x_1 + \mathbf{B}\,\eta y_1 = 1$, . . . (i)

and $\therefore$ Q (ξ, η) lies on $\quad \mathbf{A}xx_1 + \mathbf{B}yy_1 = 1$. . . . . (ii)

The polar of P, viz. (ii), is easily seen to be in the direction conjugate to CP, whose equation is $xy_1 - yx_1 = 0$.

From the symmetry of equation (i) it appears that if Q is on the polar of P, then P is on the polar of Q.

Examples.

1. Show that the tangents at the extremities of any chord intersect on the diameter conjugate to the chord.

2. Show that the polar of $(x_1 y_1)$ with reference to the parabola $y^2 = 4px$ is $yy_1 = 2p(x + x_1)$.

3. Show by transference of the origin or otherwise that

$$axx_1 + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

is the tangent at $P(x_1 y_1)$, if P is on $ax^2 + by^2 + 2gx + 2fy + c = 0$, and is the polar of P_1 if P is not on the curve.

Note. The equation $y = mx + \frac{p}{m}$ for the tangent to the parabola may be obtained as follows:

$$y = mx + \sqrt{\alpha^2 m^2 + \beta^2} \text{ touches } \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1.$$

Using the method of pp. 155-6,

$$y = m(x - \alpha) + \sqrt{\alpha^2 m^2 + \beta^2} \text{ touches } \frac{(x-\alpha)^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1,$$

i.e. $y^2 = 4px$ in limit;

$$y = mx - m\alpha + \alpha m\left(1 + \frac{\beta^2}{\alpha^2 m^2}\right)^{\frac{1}{2}}$$

$$= mx - m\alpha + \alpha m\left(1 + \frac{1}{2}\frac{2p}{\alpha m^2} + \ldots\right), \text{ by the Binomial Series,}$$

$$= mx + \frac{p}{m} - \frac{1}{2}\frac{p^2}{\alpha m^2} + \text{terms involving } \frac{1}{\alpha^2},$$

$$= mx + \frac{p}{m} \text{ in limit.}$$

As an exercise, the equation of the tangent or polar,

$$yy_1 = 2p(x + x_1),$$

should be obtained by this method of limits.

Before going back to the general equation we will obtain some further geometrical properties of the parabola and ellipse.

The Parabola.

It has already been shown that $y = mx + \frac{p}{m}$ touches $y^2 = 4px$ at $\left(\frac{p}{m^2}, \frac{2p}{m}\right)$, and that $AN = \frac{p}{m^2}$, $NP = \frac{2p}{m}$.

If this tangent meets the axis at T, $y = 0$, and $AT = x = -\frac{p}{m^2}$.

$$\therefore\ TA = AN.$$

$$\therefore\ TS = AN + AS = AN + KA = KN = SP.$$

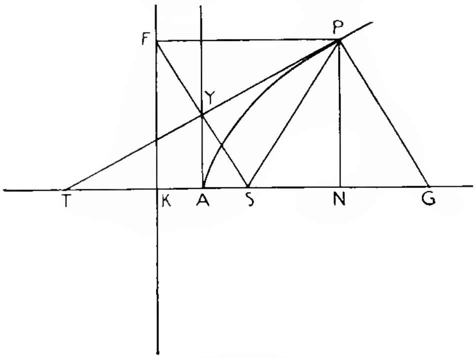


Fig. 65.

Draw PG perpendicular to the tangent at P. PG is called the *normal*.

Since $SP = ST$ and TPG is a right angle, $\therefore$ S is centre of circle TPG, and $SG = SP = TS = p\left(1 + \frac{1}{m^2}\right)$ since $TS = AN + AS$, and

$$NG = AG - AN = AS + SG - AN = p + p\left(1 + \frac{1}{m^2}\right) - \frac{p}{m^2} = 2p$$

and is constant for all normals.

NT, NG are called the subtangent and the subnormal.

Examples (*to be done by geometry or analysis*).

1. Show that the equation of PG is $m^3y + m^2(x - 2p) - p = 0$.

2. If $m = \tan\theta$, show that $\angle PSG$ is 2θ, and that PT bisects the angle between PS and PF, where PF is parallel to AX.

3. Show that the perpendicular from the focus on a tangent meets it on the tangent at the vertex.

4. Show that tangents in directions m_1, m_2 meet at T' whose co-ordinates are
$$\left(\frac{p}{m_1 m_2}, p\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\right),$$
and that this point is on the line through the middle point of the chord of contact (V) parallel to the axis.

5. The equation of a chord is $y\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = 2\left(x + \frac{p}{m_1 m_2}\right)$, where $m_1\, m_2$ give the directions of the tangents at its ends.

6. If $T'V$, in example 4, meets the curve at R, show that $T'R = RV$, and if P is the point of contact of the tangent in direction m_1, $PV^2 = 4SR \, . \, RV$, and PV is parallel to the tangent at R.

[The last result is used in the dynamical proof that the path of a particle moving under gravity in vacuo is a parabola.]

7. Show that PS meets the curve again at $P'(pm^2, -2pm)$, and that the tangents at the extremities of a focal chord (i.e. a chord of the curve passing through the focus) meet on the directrix at right angles to each other. In particular the tangents from K touch the curve at the ends of the latus rectum and make 45° with KX.

8. Show that the semi-latus rectum is the harmonic mean between two segments SP, SP' of a focal chord.

The Circle and Ellipse.

Circle. Let $x^2 + y^2 = a^2$ be the equation of a circle.

Let t_1QT be a tangent at any point $Q\,(x, y)$ meeting CX, CY at T, t_1.

Let $\angle QCT = \phi$, $x = CM = a \cos\phi$, $y = MQ = a \sin\phi$.

Then $CT = CQ \sec\phi = a \sec\phi$, and $Ct_1 = a \operatorname{cosec}\phi$.

The equation of the tangent is
$$\frac{x}{a \sec\phi} + \frac{y}{a \operatorname{cosec}\phi} = 1,$$
that is
$$x \cos\phi + y \sin\phi = a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{i})$$

$CM \, . \, CT = a^2$ and $CN \, . \, Ct_1 = a^2$, if QN is perpendicular to CY.

Take another point Q'. Let $\angle Q'CT = \phi'$.

Join QQ', and draw CL perpendicular to it. Let QQ' meet the axes at J, H.

$\angle LCQ = \frac{1}{2}(\phi' - \phi)$, and $LCX = \frac{1}{2}(\phi + \phi')$; $\therefore CL = a \cos\frac{1}{2}(\phi' - \phi)$, $CJ = CL \sec\frac{1}{2}(\phi' + \phi)$, $CH = CL \operatorname{cosec}\frac{1}{2}(\phi' + \phi)$.

The equation of QQ' is $\frac{x}{CJ} + \frac{y}{CH} = 1$,

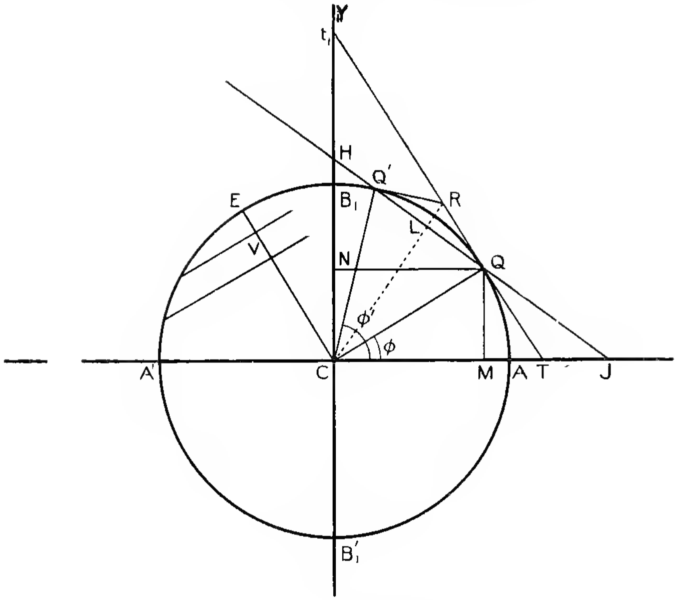


FIG. 66 *a*.

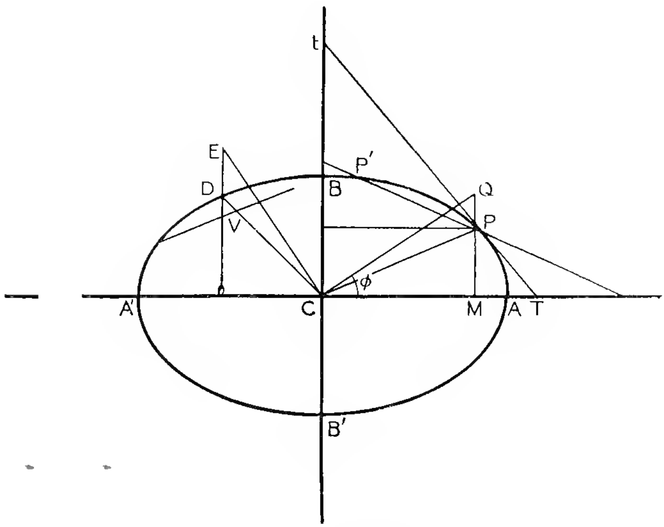


FIG. 66 *b*.

that is, $x\cos\frac{1}{2}(\phi'+\phi)+y\sin\frac{1}{2}(\phi'+\phi) = CL = \alpha\cos\frac{1}{2}(\phi'-\phi)$. (ii)

The tangent at Q' is $x\cos\phi'+y\sin\phi' = \alpha$. This meets TQ at R on CL produced. $CR = CQ\sec\frac{1}{2}(\phi'-\phi)$. By projecting CR we find the co-ordinates of R to be

$$\alpha\cos\tfrac{1}{2}(\phi+\phi')\sec\tfrac{1}{2}(\phi'-\phi),\ \alpha\sin\tfrac{1}{2}(\phi+\phi')\sec\tfrac{1}{2}(\phi'-\phi).$$

Ellipse. Now reproduce the figure with every point brought nearer the axis AA' in the ratio $\frac{\beta}{\alpha}$. If (x, y) are the co-ordinates of any point in the first figure, and (x, y') of the corresponding point in the second, $y' = \frac{\beta}{\alpha}\cdot y$, $y = \frac{\alpha}{\beta}\cdot y'$.

The lines and points in the second figure may be called the shadows of those in the first.

[If the first figure is tilted about the line AA' so as to lie on an inclined plane, till the angle of the plane is $\cos^{-1}\frac{\beta}{\alpha}$, and then its shadow is cast on the original plane by rays of light perpendicular to it, the second figure is obtained.]

If the point (x, y) is on the circle $x^2+y^2 = \alpha^2$, then x, y' is on the curve, $x^2+\left(\frac{\alpha}{\beta}y'\right)^2 = \alpha^2$, in the shadow.

i.e. $\frac{x^2}{\alpha^2}+\frac{y'^2}{\beta^2} = 1$, that is, an ellipse.

The vertices are A, A', B, B', where

$$CB = CB' = \frac{\beta}{\alpha}\text{ of } CB_1 = \frac{\beta}{\alpha}\text{ of }\alpha = \beta.$$

The angle θ (or QCM), which is *not* the same as the angle PCM, is called the *eccentric* angle of the ellipse at the point P, or the eccentric angle of P.

The co-ordinates of P are

$$CN = \alpha\cos\phi,\text{ and } NP = \frac{\beta}{\alpha}\text{ of } NQ = \beta\sin\phi.$$

The shadow of a chord is a chord, of an intersection is an intersection, of a tangent is a tangent.

The equation of the tangent at P is obtained from (i) as follows:

Any point on QT satisfies $x\cos\phi+y\sin\phi = \alpha$.

$\therefore$ any point on PT satisfies

$$x\cos\phi+\frac{\alpha}{\beta}y'\sin\phi = \alpha,\ \text{i.e. } \frac{x}{\alpha}\cos\phi+\frac{y'}{\beta}\sin\phi = 1.$$

Now suppress the ′, and we have the tangent to $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$ at a point whose eccentric angle is ϕ (generally abbreviated to 'at a point ϕ') is

$$\frac{x}{\alpha}\cos\phi+\frac{y}{\beta}\sin\phi=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)*}$$

Similarly, the chord ϕ, ϕ' is

$$\frac{x}{\alpha}\cos\tfrac{1}{2}(\phi+\phi')+\frac{y}{\beta}\sin\tfrac{1}{2}(\phi+\phi')=\cos\tfrac{1}{2}(\phi-\phi') \quad . \quad \text{(iv)*}$$

and the intersection of the tangents at ϕ, ϕ' is

$$\alpha\cos\tfrac{1}{2}(\phi'+\phi)\sec\tfrac{1}{2}(\phi'-\phi),\ \beta\sin\tfrac{1}{2}(\phi'+\phi)\sec\tfrac{1}{2}(\phi'-\phi).$$

In the ellipse, $CM\,.\,CT=\alpha^2$.

$$MP\,.\,Ct=\frac{\beta}{\alpha}\,.\,MQ\,.\,\frac{\beta}{\alpha}\,Ct_1=\frac{\beta^2}{\alpha^2}\,CN\,.\,Ct_1=\beta^2.$$

The **area** of the ellipse is area of circle $\times\ \frac{\beta}{\alpha}=\pi\alpha\beta$.

If CE is a diameter of the circle at right angles to CQ, then CE bisects all chords parallel to CQ and vice versa. CQ and CE are conjugate.

If a chord KK' is divided in any ratio at V, then its shadow kvk' is divided in an equal ratio at v, since Kk, Vv, $K'k'$ are parallel. Also parallel lines 'meet at infinity' in the one figure; their shadows still 'meet at infinity' and are parallel.

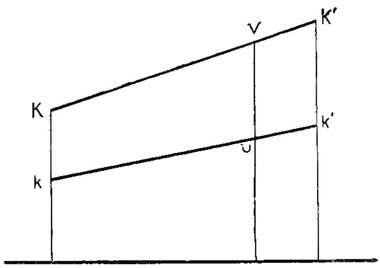


FIG. 67.

Hence conjugate or perpendicular diameters of a circle project into conjugate diameters of the ellipse. Hence if the eccentric angles of P and D are ϕ and $(\phi+90)$, then CP and CD are conjugate.

The co-ordinates of D are

$$\alpha\cos(\phi+90)=-\alpha\sin\phi,\ \text{and}\ \beta\sin(\phi+90)=\beta\cos\phi.$$

It follows at once that

$$CP^2+CD^2=\{(\alpha\cos\phi)^2+(\beta\sin\phi)^2\}+\{(-\alpha\sin\phi)^2+(\beta\cos\phi)^2\}$$
$$=\alpha^2+\beta^2.\quad \text{Compare p. 161.}$$

* These can be readily obtained also by substitution in the equations of p. 159.

The method here used is called the method of orthogonal projection. By it, properties relating to ratio, parallelism, tangency, and intersection can be passed from the circle to the ellipse. These are called projective properties. Properties relating to angles are not projective, for right angles and equal angles do not cast rectangular or equiangular shadows.

The student can develope further properties by the projective method and by the use of eccentric angles by the following series of examples.

Examples on projective method.

1. The tangent at (x_1, y_1), a point on an ellipse, is $\frac{xx_1}{\alpha^2} + \frac{yy_1}{\beta^2} = 1$.

2. If $\tan^{-1} m$ is the direction of a line in a figure, $\tan^{-1} m'$ of its shadow, $m' = \frac{\beta}{\alpha} \cdot m$.

3. The tangent to a circle in direction $\tan^{-1} m$ is

$$y = mx \pm \alpha \sqrt{1 + m^2}.$$

To an ellipse in direction $\tan^{-1} m'$

$$y = m'x \pm \sqrt{\alpha^2 m'^2 + \beta^2}.$$

4. Diameters of an ellipse in direction $\tan^{-1} m_1$, $\tan^{-1} m_2$ are conjugate if $m_1 m_2 = -\frac{\beta^2}{\alpha^2}$.

5. In Figure 62, $\frac{Q_1 V^2}{PV \cdot VP'} = \frac{CD^2}{CP^2}$, and the tangents at Q_1, Q_2 meet on CP at T, where $CV \cdot CT = CP^2$.

6. The area of the parallelogram formed by the tangents at the extremities of conjugate diameters is constant.

7. The equation of the polar of $P\ (x_1, y_1)$ is $\frac{xx_1}{\alpha^2} + \frac{yy_1}{\beta^2} = 1$, and if CP meets the polar in V and the curve in E, $CE^2 = CV \cdot CP$.

Examples on eccentric angles.

1. The equation of the normal at P is $x\alpha \sec\phi - y\beta \operatorname{cosec}\phi = \alpha^2 - \beta^2$. Hence $CG = e^2 \cdot CM$, where PG is normal, CM abscissa, and G on CA.

2. The product $SP \cdot S'P$ equals CD^2, where CP, CD are semi-conjugate diameters. $[SP = a(1 - e\cos\phi).]$

3. The normal at P bisects the angle SPS'. [$SG = ae(1 - e\cos\theta)$, if normal meets AA' in G.]

4. If SY, $S'Y'$ are perpendiculars on to a tangent $SY.S'Y' = \beta^2$. $\left[SY = \beta\sqrt{\dfrac{1 - e\cos\phi}{1 + e\cos\phi}}.\right]$

5. If the normal at P meets the axes in G, g and the diameter conjugate to CP in F, then $PF.PG = b^2$, $PF.Pg = a^2$. [PF = perpendicular from centre on tangent $= \dfrac{\alpha\beta}{\sqrt{(\beta^2\cos^2\phi + \alpha^2\sin^2\phi)}}$.]

6. The co-ordinates of the intersection of the tangents at the ends of conjugate diameters may be written $a(\cos\phi + \sin\phi)$, $b(\cos\phi - \sin\phi)$.

7. Find the co-ordinates of the intersection of the normals at (ϕ, ϕ'). $\left[\text{The abscissa is } ae^2\cos\dfrac{\phi+\phi'}{2}.\sec\dfrac{\phi-\phi'}{2}\cos\phi\cos\phi'.\right]$

An Important Property of the Hyperbola.

If $P(x_1, y_1)$ is on the hyperbola $\dfrac{x^2}{\alpha^2} - \dfrac{y^2}{\beta^2} = 1$, and PK, PL are perpendicular to the asymptotes, then $PK.PL$ is constant;

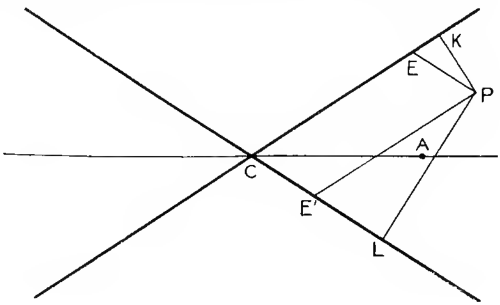


Fig. 68.

for the perpendiculars from P to $\dfrac{x}{\alpha} - \dfrac{y}{\beta} = 0$, and $\dfrac{x}{\alpha} + \dfrac{y}{\beta} = 0$ are

$$PK = \frac{\dfrac{x_1}{\alpha} - \dfrac{y_1}{\beta}}{\sqrt{\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}}} \text{ and } \frac{\dfrac{x_1}{\alpha} + \dfrac{y_1}{\beta}}{\sqrt{\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}}} = PL.$$

$$\therefore\ PK\,.\,PL = \frac{\left(\frac{x_1}{\alpha} - \frac{y_1}{\beta}\right)\left(\frac{x_1}{\alpha} + \frac{y_1}{\beta}\right)}{\frac{1}{\alpha^2} + \frac{1}{\beta^2}} = \frac{\frac{x_1^2}{\alpha^2} - \frac{y_1^2}{\beta^2}}{\frac{1}{\alpha^2} + \frac{1}{\beta^2}}$$

$$= \frac{1}{\frac{1}{\alpha^2} + \frac{1}{\beta^2}} = \frac{\alpha^2\beta^2}{\alpha^2 + \beta^2}.$$

Conversely, if a point moves so that the product of its perpendiculars from two straight lines is constant it describes a hyperbola. (See pp. 80–2 for the case of a rectangular hyperbola.)

Examples.

1. If PE, PE' are parallel to the asymptotes,

$$PE\,.\,PE' = \left(\frac{\alpha^2 + \beta^2}{4}\right) = \tfrac{1}{4}CS^2,$$

and the area of the parallelogram $PE'CE$ is $\frac{1}{2}\alpha\beta$.

$$\left(\text{Use } \tan ECA = \frac{\beta}{\alpha} \text{ and } \sin ECE' = \frac{2\alpha\beta}{\alpha^2 + \beta^2},\ \text{p. } 147.\right)$$

2. Show that, if a tangent $y = mx + \sqrt{\alpha^2 m^2 - \beta^2}$ touches at P and meets the asymptotes at R, R', then P is the middle point of RR', and the area RCR' is constant and $= \alpha\beta$.

3. The equation $y = \dfrac{a + bx + cx^2}{x + d}$ (p. 83) can be written

$$(x + d)(y - cx - b + dc) = a - bd + cd^2.$$

Show that this can be expressed as the product of two perpendiculars and therefore represents an hyperbola. First take the case $y = \dfrac{x^2 - \sqrt{2}}{x}$, then $y = \dfrac{x^2 - x}{x - 2}$, then that on p. 85.

The Auxiliary Circle.

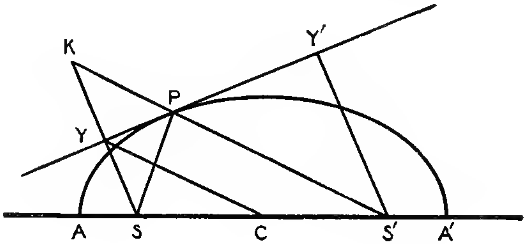
Let SY, $S'Y'$ be perpendiculars from S, S' to the tangent at any point P, viz. $\mathbf{A}xx_1 + \mathbf{B}yy_1 = 1$.

$$\frac{SY}{S'Y'} = \frac{\mathbf{A}\alpha e\,.\,x_1 - 1}{-\mathbf{A}\alpha e\,.\,x_1 - 1} = \frac{\frac{ex_1}{\alpha} - 1}{-\frac{ex_1}{\alpha} - 1} = (\text{in ellipse})\,\frac{\alpha - x_1 e}{\alpha + x_1 e} = \frac{SP}{S'P};$$

$$\text{and}\quad \frac{SY}{-S'Y'} = (\text{in hyperbola})\,\frac{ex_1 - \alpha}{ex_1 + \alpha} = \frac{SP}{S'P}.$$

Hence SP and $S'P$ are equally inclined to the tangent at P.

[If S is a luminous point, and the ellipse a reflecting rim, all the light in the plane of the ellipse is reflected to S'.]



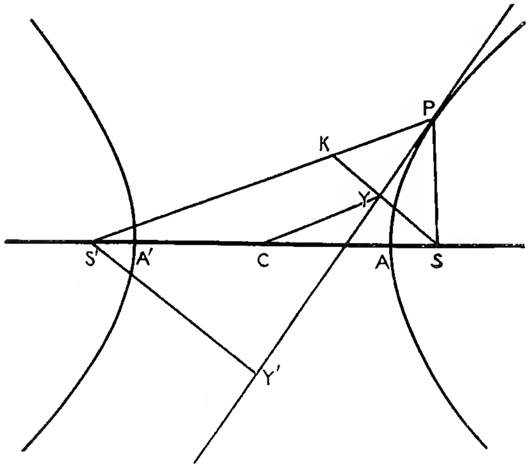


FIG. 69.

In the case of the parabola PS' becomes parallel to the axis, and the tangent bisects the angle between SP and PF (Fig. 65).

[In this case all the rays of light emanating from S are reflected in parallel lines. This gives the principle of the search-light.]

Let $S'P$ meet SY produced in K. Then $SY = YK$. Then

$$S'K = S'P \pm PK = S'P \pm SP = AA'.$$

But $CY = \frac{1}{2}S'K$, since C and Y are middle points of SS', SK.

$$\therefore\ CY = CA.$$

Hence locus of Y and of Y' is the circle $x^2 + y^2 = \alpha^2$. This is called the *auxiliary circle.*

In the parabola it degenerates into the tangent at the vertex.

It is easy to show that $SY \, . \, S'Y' = \beta^2$.

The Director Circle.

The directions of two tangents through a point ξ, η are given by the equation in m

$$\eta = m\xi + \sqrt{\frac{m^2}{\mathbf{A}} + \frac{1}{\mathbf{B}}}, \text{ i.e. } (m\xi - \eta)^2 = \frac{m^2}{\mathbf{A}} + \frac{1}{\mathbf{B}}.$$

If m_1, m_2 are the two values of m,

$$m_1 m_2 = \left(\eta^2 - \frac{1}{\mathbf{B}}\right) \div \left(\xi^2 - \frac{1}{\mathbf{A}}\right).$$

These are perpendicular if $m_1 m_2 = -1$, and hence the locus of the intersection of perpendicular tangents is the circle

$$x^2 + y^2 = \frac{1}{\mathbf{A}} + \frac{1}{\mathbf{B}} = \alpha^2 \pm \beta^2,$$

which is called the *director circle.*

For the parabola this circle degenerates into the directrix, thus—

$$(x - \alpha)^2 + y^2 = \alpha^2 + \beta^2, \; x = -p - \frac{x^2 + y^2}{2\alpha} = -p \text{ in limit.}$$

Transformation of Co-ordinates. Rotation of Axes.

Let P be any point, co-ordinates (x, y) referred to axes OX, OY.

Take another pair of rectangular axes OX_1, OY_1, where $\angle XOX_1$ = a positive angle θ.

Draw PN perpendicular to OX, PM to OX_1.

Let $OM = x'$, $MP = y'$, $ON = x$, $NP = y$.

Then ON = projection of OP on OX = sum of projections of OM, MP

$= x' \cos\theta + y' \cos(\theta + 90°)$, since MP parallel to OY_1 makes the angle $\theta + 90°$ with OX.

$$\therefore \; \mathbf{x = x' \cos\theta - y' \sin\theta}.$$

Also NP = projection of OM, MP on OY

$$= OM \sin\theta + MP \sin(\theta + 90°).$$

$$\therefore \; \mathbf{y = x' \sin\theta + y' \cos\theta}.$$

If, then, in any expression these values are written for x, y, the resulting equation in (x', y') shows the same expression referred

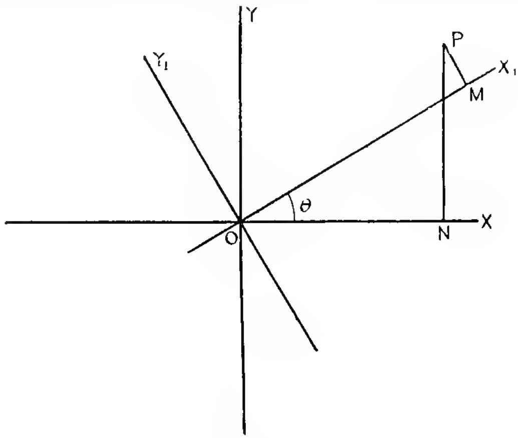


FIG. 70.

to OX_1, OY_1 as axes; and if $f(x, y) = 0$ is the equation of a locus referred to OX, OY as axes,

$$f\{(x\cos\theta - y\sin\theta),\ \ (x\sin\theta + y\cos\theta)\} = 0$$

is the equation of the same locus referred to OX_1, OY_1. (Compare p. 142.)

The following notation will be needed:

Let $A = bc - f^2$, $B = ca - g^2$, $C = ab - h^2$, $F = gh - af$, $G = hf - bg$, $H = fg - ch$, and $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$.

Then $\Delta = aA + hH + gG = hH + bB + fF = gG + fF + cC$ by direct substitution.

It is easily shown that $BC - F^2 = a\Delta$, $GH - AF = f\Delta$, &c.,

and $$ABC + 2FGH - AF^2 - BG^2 - CH^2 = \Delta^2,$$

and $$aG + hF + gC = 0 = hG + bF + fC.$$

GENERAL EQUATION OF THE SECOND DEGREE.

$$\mathbf{ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.} \quad . \quad . \quad . \quad \text{(i)}$$

Now remove the restriction that $h = 0$ that was made on p. 143. We can show by two steps of transformation that, with certain exceptions, this equation represents an ellipse (or circle) parabola, hyperbola, or pair of straight lines.

First, refer to a point $(\bar{x}, \bar{y})$, O', as origin, with axes $O'X_1$, $O'Y_1$, parallel to the original axes. O' is still to be chosen.

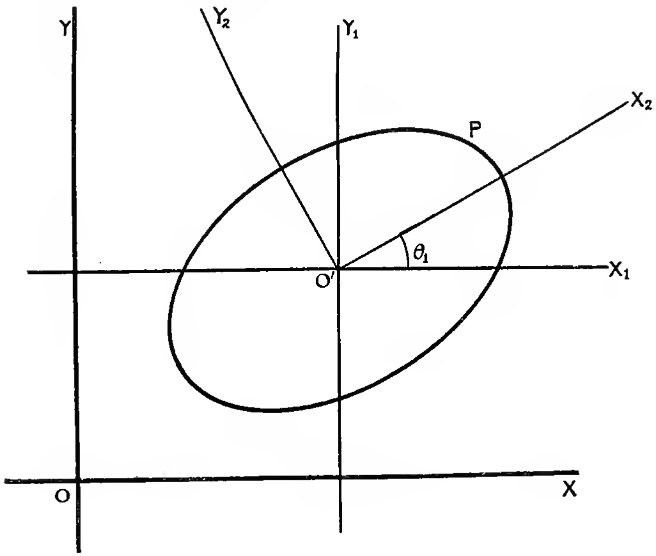


FIG. 71.

As on pp. 142, 143, if $(x' y')$ are the new co-ordinates,

$$x = x' + \bar{x}, \quad y = y' + \bar{y}.$$

Substitute and suppress the accents $''$,

$$a(x+\bar{x})^2 + 2h(x+\bar{x})(y+\bar{y}) + b(y+\bar{y})^2 + 2g(x+\bar{x}) + 2f(y+\bar{y}) + c = 0.$$

$\therefore$ by removing and rearranging brackets,

$$ax^2 + 2hxy + by^2 + 2x(a\bar{x} + h\bar{y} + g) + 2y(h\bar{x} + b\bar{y} + f) + \bar{x}(a\bar{x} + h\bar{y} + g) + \bar{y}(h\bar{x} + b\bar{y} + f) + g\bar{x} + f\bar{y} + c = 0.$$

Now choose $\bar{x}$, $\bar{y}$ so that $a\bar{x} + h\bar{y} + g = 0$,

$$h\bar{x} + b\bar{y} + f = 0;$$

we have, solving these equations,

$$\bar{x} = \frac{hf - bg}{ab - h^2} = \frac{G}{C}, \text{ and } \bar{y} = \frac{F}{C}.$$

Then the equation becomes

$$ax^2 + 2hxy + by^2 + g \cdot \frac{G}{C} + f \cdot \frac{F}{C} + c \cdot \frac{C}{C} = 0,$$

$$\text{i.e.} \quad \mathbf{ax^2 + 2hxy + by^2 + \frac{\Delta}{C} = 0.} \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

This transformation breaks down when $C = 0$, but in no other case.

O', so found, is called the *centre* of the curve. For if a point $P(x_1, y_1)$ satisfies (ii) so does $P'(-x_1, -y_1)$, and PP' is bisected at O'.

Secondly, if $C \neq 0$, rotate the axes through an angle θ (still to be chosen) with O' as origin.

The equation becomes

$$a(x\cos\theta - y\sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta)$$
$$+ b(x\sin\theta + y\cos\theta)^2 + \frac{\Delta}{C} = 0 \quad \text{(p. 174)},$$

i. e. $x^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta)$

$$+y^2(a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta)$$
$$+xy(\overline{b-a}\,2\sin\theta\cos\theta + 2h(\cos^2\theta - \sin^2\theta)) + \frac{\Delta}{C} = 0.$$

Choose θ so that the coefficient of xy is zero,

i. e. let $\quad (b-a).\sin 2\theta + 2h\cos 2\theta = 0,$

$$\mathbf{\tan 2\theta = \frac{2h}{a-b}.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

There are always two distinct real values, say θ_1 and $\theta_1 + 90°$, to satisfy this equation. For definiteness take θ_1 to be the angle between $0°$ and $+90°$ that satisfies (iii).

Give θ the value θ_1, and the equation becomes

$$\mathbf{a_1x^2 + b_1y^2 + \frac{\Delta}{C} = 0,} \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

where

$$\left.\begin{aligned} a_1 &= a\cos^2\theta_1 + 2h\cos\theta_1\sin\theta_1 + b\sin^2\theta_1 \\ b_1 &= a\sin^2\theta_1 - 2h\cos\theta_1\sin\theta_1 + b\cos^2\theta_1 \end{aligned}\right\} \quad . \quad . \quad \text{(v)}$$

$$\therefore \quad \mathbf{a_1 + b_1 = a + b,} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vi)}$$

and $\quad h(\cos^2\theta_1 - \sin^2\theta_1) + (b-a)\cos\theta_1\sin\theta_1$

$$= \text{half the coefficient of } xy = 0. \quad . \quad . \quad \text{(vii)}$$

Multiply the first of equations (v) by $\cos\theta_1$ and equation (vii) by $\sin\theta_1$, and subtract the latter.

$$a_1 \cos\theta_1 = a(\cos^3\theta_1 + \cos\theta_1 \sin^2\theta_1) + h(\cos^2\theta_1 \sin\theta_1 + \sin^2\theta_1 \sin\theta_1);$$

$$\left.\begin{aligned} \therefore\quad \mathbf{a}_1 &= \mathbf{a} + \mathbf{h}\tan\theta_1. \\ \text{Similarly,}\quad \mathbf{b}_1 &= \mathbf{b} - \mathbf{h}\tan\theta_1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (\text{viii})$$

Equations (iii) and (viii) are sufficient for the graphic construction of the curve of equation (ii), but it is important to discriminate between the curves included without solving.

We have $\quad a_1 b_1 = ab + h(b-a)\tan\theta_1 - h^2\tan^2\theta_1$.

But $\quad (b-a)\tan\theta_1 + h(1-\tan^2\theta_1) = 0$ from (vii).

$$\therefore\ \mathbf{a}_1\mathbf{b}_1 = ab - h^2(1-\tan^2\theta_1) - h^2\tan^2\theta_1 = ab - h^2 = \mathbf{C}. \quad . \quad (\text{ix})$$

From (vi) and (ix) a_1, b_1 are roots of the equation

$$X^2 - (a+b)X + C = 0.$$

Case I. *a_1, b_1 and $-\frac{\Delta}{C}$ all the same sign.*

Let $$\frac{a_1}{-\dfrac{\Delta}{C}} = \frac{1}{\alpha^2}, \quad \frac{b_1}{-\dfrac{\Delta}{C}} = \frac{1}{\beta^2}.$$

Then α and β are real.

Equation (iv) becomes $\dfrac{X^2}{\alpha^2} + \dfrac{Y^2}{\beta^2} = 1$, and the locus is an *ellipse.*

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = -\frac{C}{\Delta}(a_1+b_1) = -\frac{C}{\Delta}(a+b) \text{ from (vi)},$$

and $$\frac{1}{\alpha^2\beta^2} = \frac{a_1 b_1 C^2}{\Delta^2} = \frac{C^3}{\Delta^2} \text{ from (ix)}.$$

Therefore α^2 and β^2 are the roots in r^2 of

$$\frac{1}{\mathbf{r}^4} + \frac{\mathbf{C}}{\Delta}(\mathbf{a}+\mathbf{b})\frac{1}{\mathbf{r}^2} + \frac{\mathbf{C}^3}{\Delta^2} = 0. \quad . \quad . \quad . \quad (\text{x})$$

The roots are equal if $4C = (a+b)^2$, i.e. if $4h^2 + (a-b)^2 = 0$, if $h = 0$ and $a = b$. This is the condition for a circle.

Case II. *a_1 and b_1 of different signs.* Suppose a_1 and $-\frac{\Delta}{C}$ of the same sign.

Let $$\frac{a_1}{-\dfrac{\Delta}{C}} = \frac{1}{\alpha^2}, \quad \frac{b_1}{\dfrac{\Delta}{C}} = \frac{1}{\beta^2};$$ then α and β are real.

Equation (iv) becomes $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$, and the locus is a *hyperbola.*

[If b_1 and $-\frac{\Delta}{C}$ were of the same sign, the equation could be written $-\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1.$]

$$\frac{1}{\alpha^2} + \left(-\frac{1}{\beta^2}\right) = -\frac{C}{\Delta}(a+b),$$

and
$$\frac{1}{\alpha^2} \times \left(-\frac{1}{\beta^2}\right) = \frac{a_1 b_1 C^2}{\Delta} = \frac{C^3}{\Delta^2};$$

α^2 and $-\beta^2$ are then the roots in r^2 of equation (x).

$\alpha^2 = \beta^2$, if $a+b=0$. This is the condition for a rectangular hyperbola.

Case I, which leads to the equation of an ellipse, is obtained when a_1, b_1 are of the same sign, that is, when $C\,(= ab - h^2)$ is positive, since $a_1 b_1 = C$; that is, when $ab > h^2$.

We must have the further condition that a_1 and Δ are of opposite signs. Otherwise equation (iv) gives the sum of positive quantities $= 0$ which is not satisfied by any real values of x and y. From equation (vi) a_1 and a are of the same sign, since a_1, b_1 are the same and also a, b, since $ab > h^2$.

The area of an ellipse is $\pi\alpha\beta$ (p. 169) $= \pi\Delta \div -C\sqrt{C}$.

Case II, which leads to the equation of an hyperbola, is obtained when $ab < h^2$.

Equation (x) gives the lengths of the semi-axes.

Rewriting (iv) as $\dfrac{x^2}{-\dfrac{\Delta}{Ca_1}} + \dfrac{y^2}{-\dfrac{\Delta}{Cb_1}} = 1$, we see that

$$\alpha = \sqrt{-\frac{\Delta}{Ca_1}}, \quad \beta = \sqrt{-\frac{\Delta}{Cb_1}}$$

are to be measured along $O'X_1$, $O'Y_1$ respectively. Equations (viii) then distinguish between α and β.

The equations of the axes of the curve referred to $O'X_1$, $O'Y_1$

are $y = x \tan \theta_1$, $y = x \tan(\theta_1 + 90) = -x \cot \theta_1$. Points on these all satisfy the equation

$$(y - x \tan \theta_1)(y + x \cot \theta_1) = 0,$$

i.e. $y^2 - x^2 = xy(\tan \theta_1 - \cot \theta_1) = xy \cdot \dfrac{\sin^2 \theta_1 - \cos^2 \theta_1}{\sin \theta_1 \cos \theta_1} = xy \cdot \dfrac{b-a}{h}$;

i.e. $\dfrac{x^2 - y^2}{a-b} = \dfrac{xy}{h}$, which is therefore the equation of the axes referred to $O'X_1$, $O'Y_1$.

E.g. draw the curve $2x^2 - 4xy + 5y^2 + 6x - 4y = 0$,

$$a = 2,\ b = 5,\ c = 0,\ f = -2,\ g = 3,\ h = -2,$$
$$C = 6,\ G = -11,\ F = -2,\ \Delta = gG + fF + cC = -29.$$

The centre is $\dfrac{G}{C} = -\frac{11}{6}$, $\dfrac{F}{C} = -\frac{1}{3}$.

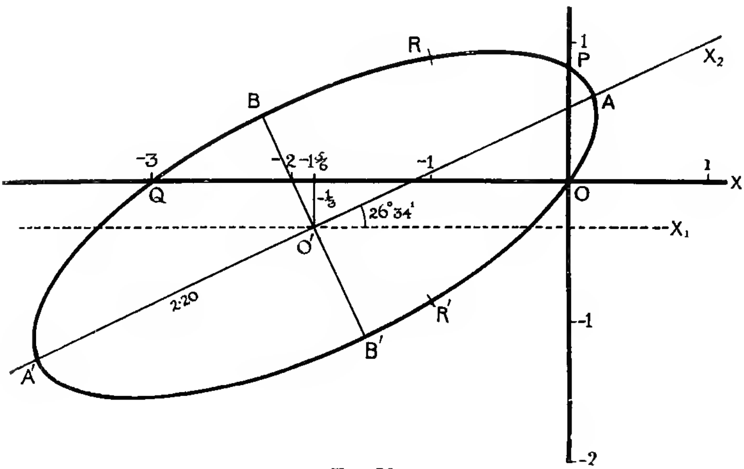


FIG. 72.

The equation referred to the centre is

$$2x^2 - 4xy + 5y^2 = -\frac{\Delta}{C} = \tfrac{29}{6}.$$

$$\frac{2\tan\theta}{1 - \tan^2\theta} = \tan 2\theta = \frac{2h}{a-b} = \tfrac{4}{3},\ \tan\theta = \tfrac{1}{2} \text{ or } -2,$$

$$\theta_1 = \tan^{-1}\tfrac{1}{2} = 26^\circ\, 34' \text{ approx.}$$

$$a_1 = a + h\tan\theta_1 = 1,\ b_1 = 6.$$

$$\alpha^2 = -\frac{\Delta}{Ca_1} = \tfrac{29}{6},\ \alpha = 2{\cdot}20\ldots,\ \beta^2 = \tfrac{29}{36},\ \beta = {\cdot}90\ldots.$$

The work can be checked from the drawing by special values.

Thus in the original equation, if $x = 0$, $y = 0$ or $\frac{4}{5}$ (P),
$y = 0$, $x = -3$ (Q).

If $x = -1$, $5y^2 = 4$, $y = \pm{\cdot}89$ (RR').

If $\Delta = 0$, equation (x), i.e. $C^3 . r^4 + C\Delta(a+b)r^2 + \Delta^2 = 0$, has both its roots zero, and the curve is neither ellipse nor hyperbola. (iv) becomes $a_1 x^2 + b_1 y^2 = 0$. If a_1 and b_1 are of the same sign, i.e. if $ab > h^2$, from (ix) the only solution is $x = 0 = y$; i.e. the point O'.

If $ab < h^2$, a_1 and b_1 are of opposite signs, and we have

$$x = \sqrt{-\frac{b_1}{a_1}} \cdot y \text{ or } x = -\sqrt{-\frac{b_1}{a_1}} \cdot y,$$

i.e. a pair of straight lines.

If then $\Delta = 0$ and $C < 0$, the general equation represents a pair of straight lines.

[Ex. Show that if $\Delta = 0 = C$, the equation represents two parallel lines.]

Now return to the case omitted, where $C = 0$, $h^2 = ab$.

a and b are of the same sign. Write the equation so that a, and $\therefore b$, is positive. Take $a = l^2$, $b = m^2$, and $\therefore h = lm$.

The equation becomes

$$l^2x^2 + 2lmxy + m^2y^2 + 2gx + 2fy + c = 0,$$

i.e.
$$(lx + my)^2 + 2gx + 2fy + c = 0.$$

We cannot make the first transference to a centre, but we can remove the xy as before by rotating the axes through θ, where

$$\tan 2\theta = \frac{2h}{a-b} = \frac{2lm}{l^2 - m^2} = \frac{2\dfrac{m}{l}}{1 - \dfrac{m^2}{l^2}}.$$

$\theta_1 = \tan^{-1}\dfrac{m}{l}$ satisfies this equation (p. 58).

Write $n^2 = m^2 + l^2$. Then $\cos\theta = \dfrac{l}{n}$, $\sin\theta = \dfrac{m}{n}$.

Hence in the rotation we must write $\dfrac{lx-my}{n}$ for x and $\dfrac{mx+ly}{n}$ for y, and therefore $\dfrac{l^2x+m^2x}{n} = nx$ for $lx+my$.

The equation becomes $n^2x^2+2g\dfrac{lx-my}{n}+2f\dfrac{mx+ly}{n}+c = 0$,

i. e. $$n^3x^2+2x(gl+fm)+2y(fl-gm)+cn = 0.$$

This is the equation of a parabola (p. 149) whose axis is parallel to OY_1, the new axis of y, which is $y = x\tan(\theta_1+90)$, i. e. $lx+my = 0$ referred to the old axis.

SUMMARY. The general equation of the 2nd degree represents:

if $\Delta = 0$, and $C \not> 0$, a pair of straight lines;

if $\Delta = 0$, and $C > 0$, one real point;

if $\Delta \neq 0$, and $C < 0$, an hyperbola;

if $\Delta \neq 0$, and $C = 0$, a parabola;

if $\Delta \neq 0$, $C > 0$, and a and Δ are of different signs, an ellipse.

if $\Delta \neq 0$, $C > 0$, and a and Δ are of the same sign, no real points.

EXAMPLES.

1. Find, where possible, the centre and the equations and lengths of the axes of the following curves and draw them. Verify by finding where the curves intersect the axes of co-ordinates (by putting $x = 0$ and $y = 0$ successively in the original equations). If the curve is a parabola, find the vertex.

(i) $3x^2-7xy+15y^2+24x = 0$.

(ii) $4x^2+18xy-7y^2+12x+7y+15 = 0$.

(iii) $3x^2-7xy+8x+4 = 0$.

(iv) $4x^2+12xy+9y^2-10x-2y+5 = 0$.

(v) $6x^2-xy-12y^2-4x+23y-10 = 0$.

2. Remove, by transference of axes, the x^2 term from

$$2x^3+6x^2+7x-8 = y,$$

the x term from $2x^2+3x-4 = y$, and the xy term and absolute term from $x^2-2xy+y^2+3x-4 = 0$.

3. Show that if $A = B = C = 0$, then $\Delta = 0$, and the general equation represents two coincident straight lines.

Intersections of General Equations of First and Second Degree.

$y = mx + k$ intersects $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at points whose abscissae are given by substituting $mx + k$ for y in the second equation.

The result is

$$x^2 (a + 2hm + bm^2) + 2x \{k(h + mb) + g + mf\} + bk^2 + 2fk + c = 0.$$

If the intersections are $P(x_1 y_1)$ and $P'(x_2 y_2)$ and x', y' is the middle point of PP', $\quad y' = mx' + k,$

and $$x' = \frac{x_1 + x_2}{2} = -\frac{k(h + mb) + g + mf}{a + 2hm + bm^2}.$$

Eliminating k, we find the locus of the middle point of parallel chords to be

$$ax + hy + g + m(hx + by + f) = 0.$$

This is a straight line through the centre of the curve (p. 176).

Its direction is $\tan^{-1} m'$, where $m' = -\dfrac{a + mh}{h + bm}$;

$$\therefore\ a + h(m + m') + bmm' = 0. \quad . \quad . \quad . \quad . \qquad \text{(xi)}$$

This relation is symmetrical and we have the general property of conjugate diameters. (Compare p. 160.)

One root is infinite if $a + 2hm + bm^2 = 0$. This equation gives the directions of the asymptotes of the hyperbola (where $h^2 > ab$). We know from p. 148 that the asymptotes pass through the centre.

Their equations are therefore $y - \dfrac{F}{C} = m\left(x - \dfrac{G}{C}\right)$, where $a + 2hm + bm^2 = 0$. Eliminate m.

$$a(Cx - G)^2 + 2h(Cx - G)(Cy - F) + b(Cy - F)^2 = 0,$$

$$C^2(ax^2 + 2hxy + by^2) - 2Cx(aG + hF) - 2Cy(hG + bF)$$
$$+ aG^2 + 2hFG + bF^2 = 0.$$

Now $aG + hF = -gC$, and $hG + bF = -fC$ from p. 175;

$$\therefore\ C^2(ax^2 + 2hxy + by^2 + 2gx + 2fy + c)$$
$$= cC^2 - G(aG + hF) - F(hG + bF) = C(cC + gG + fF) = C\Delta.$$

The asymptotes are therefore

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \frac{\Delta}{C} = 0. \quad . \quad . \quad \text{(xii)}$$

Hence to obtain the asymptotes we have only to add (or subtract) a constant to the equation. Obviously this constant can be determined by making it satisfy the condition for a pair of straight lines.

E.g. the asymptotes of

$$xy + 2y^2 - 4x = 0 \text{ are } xy + 2y^2 - 4x + d = 0,$$

where

$$0 = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 + 0 + 0 - 2(-2)^2 - d(\tfrac{1}{2})^2,$$

$$d = -32,$$

and the factors are $(y-4)(x+2y+8) = 0$.

Thus all hyperbolas whose equations differ only by a constant have the same asymptotes. They have obviously the same centres since c is not involved in the equations for the coordinates of the centre. They have also the same directions for conjugates since c does not enter in equation (xi), p. 183. The asymptotes themselves are included in this 'family' of curves.

An hyperbola and its conjugate are particular cases.

[In the standard form the hyperbola is $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} - 1 = 0$, the asymptotes $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 0$, and the conjugate $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} + 1 = 0$.*]

If, then, any line be drawn to meet a family of curves at $P_1 P_2$, $Q_1 Q_2$, $R_1 R_2$, &c., and their common asymptotes at $L_1 L_2$, and the conjugate line be drawn to cut this at V, then V is the middle point of $P_1 P_2$, $Q_1 Q_2$, $R_1 R_2$, ... and $L_1 L_2$, so that $P_1 L_1 = P_2 L_2$, &c. That the point of contact of a tangent is the middle point of the part intercepted between the asymptotes is a particular case. (Fig. 63.)

* The equation of the conjugate to the general curve is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - 2\frac{\Delta}{C} = 0,$$

for the sum of the expressions on the left hand of the equations of the two hyperbolas is twice that of the equation of the asymptotes in the standard form, and this equality is not affected by change of axes. The Δ's for the two curves are equal and opposite, and the roots of r^2 in equation (x) are changed in sign.

EXAMPLE. Draw on the same figure $2x^2 - 3xy - 2y^2 = 4$,

$$2x^2 - 3xy - 2y^2 = -4, \text{ and } 2x^2 - 3xy - 2y^2 = 0.$$

Draw $y = 8x + 6$ to intersect these at $P_1 P_2$, $Q_1 Q_2$, and $R_1 R_2$, and find the equation of the conjugate diameter. Observe that

$$P_1 R_1 = P_2 R_2, \text{ and } Q_1 R_1 = Q_2 R_2.$$

If $x_1 y_1$ is a fixed point J, then a line through it in direction θ may be written

$$y - y_1 = (x - x_1)\tan\theta, \text{ i.e.}$$

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r,$$

where $r = JP$, P being (xy)

and $x = x_1 + r\cos\theta$,

$$y = y_1 + r\sin\theta.$$

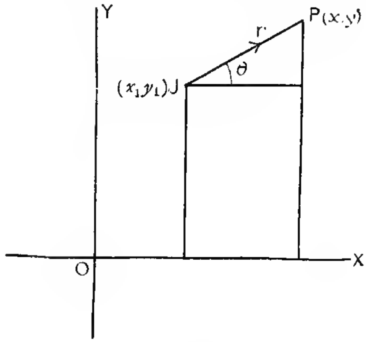


FIG. 73.

If a line JPP' be drawn to meet the curve

$$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

the distances

$$JP\ (= r_1),\ JP'\ (= r_2)$$

are the roots in r of

$$a(x_1 + r\cos\theta)^2 + 2h(x_1 + r\cos\theta)(y_1 + r\sin\theta) + b(y_1 + r\sin\theta)^2 + 2g(x_1 + r\cos\theta) + 2f(y_1 + r\sin\theta) + c = 0,$$

i.e. $(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta)r^2$

$$+ 2r[\cos\theta(ax_1 + hy_1 + g) + \sin\theta(hx_1 + by_1 + f)] + f(x_1 y_1) = 0, \quad \text{(xiii)}$$

where the last term is to be obtained by writing $(x_1 y_1)$ in the left-hand side of the equation of the curve.

$$\text{Hence } JP \cdot JP' = r_1 r_2 = \frac{f(x_1 y_1)}{a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta}.$$

If a second line JQQ' is drawn in the direction θ_1,

$$\frac{JQ \cdot JQ'}{JP \cdot JP'} = \frac{f(x_1 y_1)}{a\cos^2\theta_1 + 2h\cos\theta_1\sin\theta_1 + b\sin^2\theta_1}$$

$$\div \frac{f(x_1 y_1)}{a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta}$$

$$= \frac{a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta}{a\cos^2\theta_1 + 2h\cos\theta_1\sin\theta_1 + b\sin^2\theta_1},$$

and does not depend on the position of J at all.

This is one of the most general properties in the geometry of the conic section.

Thus if J were taken at C, the centre of the curve, and diameters RCR', TCT' were drawn in directions θ, θ_1,

$$\frac{JQ \, . \, JQ'}{JP \, . \, JP'} = \frac{CT \, . \, CT'}{CR \, . \, CR'} = \frac{CT^2}{CR^2}. \quad . \quad . \quad . \quad . \qquad \text{(xiv)}$$

The following examples suggest particular cases, and also show what a variety of results are deducible from equation (xiii).

Examples.

1. The coefficient of r disappears for all values of θ if

$$ax_1 + hy_1 + g = 0 \text{ and } hx_1 + by_1 + f = 0;$$

the values of r are equal and opposite; then $x_1 \, y_1$ is the centre.

2. One root of r is infinite if

$$a \cos^2 \theta + 2h \sin \theta \cos \theta + b \sin^2 \theta = 0,$$

and both roots are infinite if in addition $(x_1 \, y_1)$ is the centre. (The asymptotes.)

3. If $(x_1 \, y_1)$ is on the curve, the absolute term is zero. Both roots are zero, and the line $\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta}$ is a tangent, if $\tan \theta = - \frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f}$. The equation of the line then reduces to

$$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

If $(x_1 \, y_1)$ is not on the curve, the last written equation (as on pp. 163–4) represents the polar of $(x_1 \, y_1)$.

4. If a line through J meets the curve in P, P' and the polar of J in R, then $\frac{2}{JR} = \frac{1}{JP} + \frac{1}{JP'}$.

5. As a particular case of equation (xiv), show that

$$PM^2 : AM \, . \, MA' :: \beta^2 : \alpha^2 \text{ (p. 146)},$$

and $$QV^2 : PV \, . \, VP' :: CD^2 : CP^2 \text{ (p. 170, Ex. 5)}$$

for all central conics.

6. Find the co-ordinates of the centre of a conic by finding the pole of a line infinitely distant from the centre.

Polar Co-ordinates.

It is not within the scheme of this book to deal with polar co-ordinates at length.

The position of a point P in a plane is defined if the angle θ, which OP (drawn from a fixed origin O) makes with a fixed initial line OX, is known and the length OP (r) is known. θ is the *vectorial angle*, r the *radius vector*. θ may have any value, positive or negative. r, θ are called the polar co-ordinates of P. O is called the *origin* or *pole*.

The co-ordinates, x, y, hitherto used, are called *Cartesian*

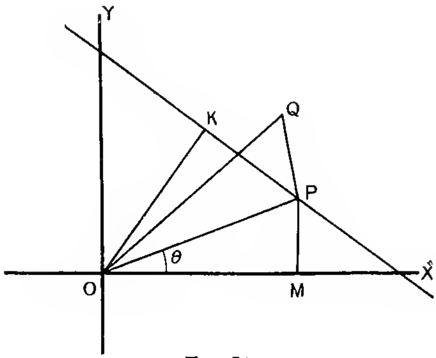


FIG. 74.

co-ordinates.* It is evident that for the same point $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2+y^2$, $\theta = \tan^{-1}\frac{y}{x}$. We can thus readily pass from one system to the other.

It is a matter simply of convenience which system shall be used.

Equation of a Straight Line. If OK is perpendicular to any line PK through P (r, θ), and if the co-ordinates (p, δ) of K are given, then $\cos(\delta \backsim \theta)† = \frac{OK}{OP} = \frac{p}{r}$.

The required equation is $r\cos(\delta \backsim \theta) = p$.

[In Cartesians this is $x\cos\delta + y\sin\delta = p$, which should be compared with p. 138.]

The distance d between two points P, Q $(r_1\,\theta_1)$, $(r_2\,\theta_2)$ is given by

$$PQ^2 = OP^2 + OQ^2 - 2\,OP.\,OQ\cos(\theta_1 \backsim \theta_2) \quad \text{(p. 53)}$$

$$d^2 = r_1{}^2 + r_2{}^2 - 2\,r_1\,r_2\cos(\theta_1 \backsim \theta_2).$$

* From Descartes, who originated them in the seventeenth century.

† $\sim$ means 'difference between', the greater to be taken first.

[Formula [A], p. 131, can be obtained by substitution.]

The area of a triangle can be obtained by substitution from that of p. 139.

The result is

$$\Delta = \tfrac{1}{2}\{r_1 r_2 \sin(\theta_1 - \theta_2) + r_2 r_3 \sin(\theta_2 - \theta_3) + r_3 r_1 \sin(\theta_3 - \theta_1)\}.$$

If $(r_3\, \theta_3)$ is on the line $P\,(r_1\, \theta_1)$, $Q\,(r_2\, \theta_2)$ this area is 0.

Hence the condition that r, θ is on the line PQ is

$$0 = \sin(\theta_1 - \theta_2) + \frac{r}{r_1}\sin(\theta_2 - \theta_3) + \frac{r}{r_2}\sin(\theta - \theta_1).$$

This is therefore the equation of the line PQ.

The equation of the circle, centre $C\ (c, \delta)$, radius ρ (Fig. 75), is

$$\begin{aligned}\rho^2 = CP^2 &= OC^2 + OP^2 - 2\,OC\,.\,OP\cos(\theta - \delta)\\ &= c^2 + r^2 - 2cr\cos(\theta - \delta).\end{aligned}$$

If O is on the circle, $c^2 = \rho^2$, and the equation is

$$r = 2\rho\cos(\theta - \delta),$$

and if the centre is on the initial line,

$$\delta = 0, \text{ and } r = 2\rho\cos\theta.$$

The equation of an ellipse or hyperbola, with the centre as

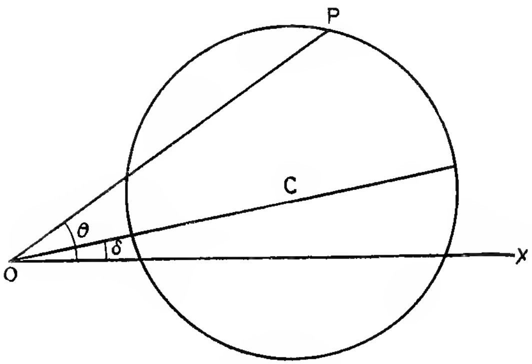


Fig. 75.

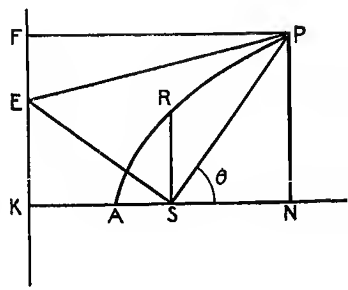


Fig. 76.

pole and the transverse axis as initial line, is, by substitution in

$$\mathbf{A}x^2 + \mathbf{B}y^2 = 1,$$

$$r^2(\mathbf{A}\cos^2\theta + \mathbf{B}\sin^2\theta) = 1.$$

The focus is, however, the more useful pole.

$$SP = e \, . \, PF = e\,(KS + SN).$$

$$r = e\,(KS + r\cos\theta).$$

Let $e \, . \, KS = l$. Then l is the semi-latus rectum, SR in Fig. 76.

The equation is then $\dfrac{l}{r} = 1 - e\cos\theta$.

For the parabola this is $\dfrac{l}{r} = \sin^2\dfrac{\theta}{2}$.

Let $P\ (r_1\,\theta_1)$, $P'\ (r_2\,\theta_2)$ be on the conic $\dfrac{l}{r} = 1 - e\cos\theta$.

The equation of the line joining these points is

$$\frac{1}{r}\sin(\theta_1 - \theta_2) = \frac{1}{r_1}\sin(\theta - \theta_2) - \frac{1}{r_2}\sin(\theta - \theta_1), \text{ p. 188 };$$

$$\therefore \quad \frac{l}{r}\sin(\theta_1 - \theta_2) = (1 - e\cos\theta_1)\sin(\theta - \theta_2) - (1 - e\cos\theta_2)\sin(\theta - \theta_1)$$

$$= 2\sin\tfrac{1}{2}(\theta_1 - \theta_2)\cos\{\theta - \tfrac{1}{2}(\theta_1 + \theta_2)\} - e\cos\theta\sin(\theta_1 - \theta_2).$$

Hence the equation of PP' is

$$\frac{l}{r} = \sec\tfrac{1}{2}(\theta_1 - \theta_2)\cos\{\theta - \tfrac{1}{2}(\theta_1 + \theta_2)\} - e\cos\theta.$$

Let θ_2 approach and ultimately coincide with θ_1 (as in the argument of p. 159).

Then the equation of the tangent at P is

$$\frac{l}{r} = \cos(\theta - \theta_1) - e\cos\theta.$$

Examples.

1. If the tangent at P meets the directrix at F, show that PSF is a right angle.

2. The intersection of the tangents at α and β has radius vector $l \div \{\cos\frac{1}{2}(\alpha - \beta) - e\cos\frac{1}{2}(\alpha + \beta)\}$ and vectorial angle $\frac{1}{2}(\alpha + \beta)$.

3. If tangents at P, Q meet in T, ST bisects the angle QSP. In the parabola $ST^2 = SP \, . \, SQ$ and the triangles SPT, STQ are similar.

4. If three tangents to a parabola intersect at T_1, T_2, T_3, then a circle can be drawn through S, T_1, T_2, T_3.

5. Express $ax + by = c$ in polar co-ordinates, by writing $\dfrac{b}{a} = \tan\alpha$ (see p. 141, Ex. 1).

The following curves, which are not readily investigated in Cartesian co-ordinates, are easily traced.

1. $r = a + b\cos\theta$; $a \neq b$ (limaçon); if $a = b$, this becomes $r = 2a\cos^2\frac{1}{2}\theta$ (cardioid).

2. $r = a\theta$ (spiral).

3. $\log r = a\theta$ (logarithmic or equiangular spiral).

4. $r^2 = a^2\cos 2\theta$ (lemniscate).

5. $r(\cos^3\theta + \sin^3\theta) = 3a\cos\theta\sin\theta$ (folium of Descartes).

6. $r = a\cos 2\theta$, $a\sin 2\theta$, $a\cos 3\theta$, $a\sin 3\theta$, and generally $a\cos n\theta$, n integral.

E.g. To trace the cardioid $r = 2a\cos^2\frac{1}{2}\theta$.

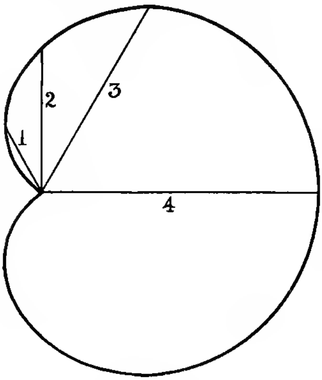


Fig. 77.

θ	r
0	$4a$
$\pm 60°$	$3a$
90	$2a$
120	$1a$
180	0

A general discussion of plane curves of higher degree than the second can be carried out most advantageously by means of the Calculus.

Note. *Oblique Co-ordinates.* If, in Figure 48, p. 131, YY' is not at right angles to $X'X$, we can still draw PM parallel to YO and define the position of P by the co-ordinates $x = OM$, $y = MP$. The axes are then said to be oblique. It is easy to show that formulae and methods not involving angles or absolute lengths, but involving ratios of lengths, intersections, and tangency, are unaffected, and that the equation of the first degree still represents a straight line. It can be shown that the equation of the second degree still represents a conic section. In particular (from Ex. 5, p. 170), $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$ represents an ellipse referred to a pair of conjugate diameters as axes of co-ordinates.

The use of oblique co-ordinates often simplifies and often generalizes problems and propositions. They have not been used in this section to avoid confusion.

SECTION VII

DIFFERENTIAL AND INTEGRAL CALCULUS

In very large regions both of pure and applied mathematics it is necessary to deal with quantities so small as to be less than any assigned finite quantity. This is the case, for example, when we wish to find the laws which determine the formation of a curve, or to determine the area of a curve, or to discuss the motion of a body whose velocity is continually changing. We cannot deal with such quantities directly by the rules of finite algebra, but by using the method of limits we can obtain finite ratios of vanishingly small quantities and operate on these.

Let $y = f(x)$ be any function which can be represented by a graph drawn by a pencil that does not break contact with the paper. Let AB be part of the graph. Let y and $y + \delta y$ be the values of the function corresponding to values of the variable x and $x + \delta x$, where δy and δx are finite quantities, which are presently taken to be small.

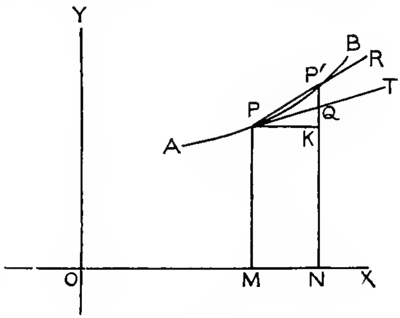


FIG. 78.

Let P be (x, y) and P' $(x + \delta x, y + \delta y)$. Then in the figure MP is y, NP' is $y + \delta y$, OM is x, ON is $x + \delta x$; MN and KP' are δx and δy, where PK is parallel to OX.

Join PP' and produce to any point R. Let $\angle P'PK = \phi$.

Then $\tan \phi = \dfrac{\delta y}{\delta x}$.

Now diminish δx, so that N approaches M and P' approaches P. As NM becomes indefinitely small, in general KP' becomes indefinitely small, but $\tan\phi$ in general remains finite, and the line PR takes a definite position, say PT. Let $\angle TPK = \theta$.

$$\text{Then } \tan\theta = \mathop{\mathrm{Lt}}_{\delta x \to 0} \tan\phi = \mathop{\mathrm{Lt}}_{\delta x \to 0} \frac{\delta y}{\delta x} = \mathop{\mathrm{Lt}}_{\delta x \to 0} \frac{(y+\delta y)-y}{(x+\delta x)-x}$$

$$= \mathop{\mathrm{Lt}}_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{(x+\delta x)-x}.$$

When the quantity last written can be evaluated and gives a determinate result, the result is called the '*derived function* of $f(x)$', or 'the *differential coefficient* of y (or $f(x)$) with respect to x', and is written $D_x y$ or $f'(x)$.* The process is called *differentiation*. It does not depend on the graphic representation of the function.

$$\text{DEFINITION.} \quad f'(x) = D_x y = \mathop{\mathrm{Lt}}_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x},$$

when $y = f(x)$. δx and δy are spoken of as the *increments* of x and y.

The values of this limit for several functions were found on pp. 128–9, where h is written throughout instead of δx. The following table shows these and some others.

Standard Forms.

Function.	Derived Function.	Function.	Derived Function.
i. x^n	nx^{n-1}	vii. $\sin x$	$\cos x$
ii. e^x	e^x	viii. $\cos x$	$-\sin x$
iii. a^x	$a^x \times \log_e a$	ix. $\tan x$	$\sec^2 x$
iv. $\log_e x$	$\frac{1}{x}$	x. $\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
v. $\log_a x$	$\frac{1}{x} \times \log_a e$	xi. $\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
vi. kx	k	xii. $\tan^{-1} x$	$\frac{1}{1+x^2}$

In vii–xii x is in radian measure.

* It is also commonly written $\frac{dy}{dx}$, but this notation is not needed in the present section.

Nos. x to xii may be proved as follows or as on p. 199.

$$y = \sin^{-1}x. \quad \therefore \quad x = \sin y, \; x + \delta x = \sin(y + \delta y).$$

$$\therefore \quad \delta x = \sin(y+\delta y) - \sin y = 2\cos(y + \tfrac{1}{2}\delta y) \,.\, \sin(\tfrac{1}{2}\delta y) \quad \text{(p. 57)}.$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{\frac{1}{2}\delta y}{\sin(\frac{1}{2}\delta y)} \cdot \frac{1}{\cos(y + \frac{1}{2}\delta y)}.$$

$$\therefore \quad D_x y = \underset{\delta x \to 0}{\mathrm{L}^t} \frac{\delta y}{\delta x} = \underset{\delta y \to 0}{\mathrm{L}^t} \frac{\frac{1}{2}\delta y}{\sin(\frac{1}{2}\delta y)} \times \underset{\delta y \to 0}{\mathrm{L}^t} \frac{1}{\cos(y + \frac{1}{2}\delta y)} \quad \text{(p. 104)},$$

since $\delta y = 0$, when $\delta x = 0$

$$= 1 \times \frac{1}{\cos y} \text{ (p. 103)} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}.$$

Similarly for $\cos^{-1}x$.

For $y = \tan^{-1}x$, $\delta x = \tan(y+\delta y) - \tan y = \dfrac{\sin \delta y}{\cos y \cos(y+\delta y)}$

(p. 57).

$$D_x y = \mathrm{L}^t \frac{\delta y}{\delta x} = \mathrm{L}^t \frac{\delta y}{\sin \delta y} \,.\, \cos y \,.\, \cos(y + \delta y)$$

$$= 1 \times \cos y \times \cos y = \frac{1}{\sec^2 y} = \frac{1}{1 + x^2}.$$

The limiting position of PR in the figure on p. 191 is defined to be the *tangent* to the curve at the point P. The process of finding it is equivalent to that in Co-ordinate Geometry (p. 156), where a line $y = mx + k$ meets a curve in two points whose abscissae may be written x and $x + \delta x$, and the value of m is found when the roots are equal, i.e. when $\delta x = 0$.

Thus $f'(x) = \tan\theta$, where θ is the inclination to OX of the tangent to the curve at (x, y). $\tan\theta$ is called the *gradient* of the curve.

The relation between the direction or gradient of a curve and its derived function is best realized from an example.

Take the curve $y = \sin x$, where x is in radian measure, and below it draw the curve $y = f'(x) = \cos x$ (Standard Form vii).

The value of $f'(x)$ for any value of x = gradient of $\sin x$ for the same value of x.

x Radians	$f(x)$ $= \sin x$	$f'(x) = \cos x$ $= \tan\theta$	Hence θ
0	0	1	45°
$\frac{1}{6}\pi$	·5	·866	40·9°
$\frac{1}{3}\pi$	·866	·5	26·6°
$\frac{1}{2}\pi$	1·	0	0
$\frac{2}{3}\pi$	·866	−·5	−26·6°
$\frac{5}{6}\pi$	·5	−·866	−40·9°
π	0	−1·	−45°
$\frac{7}{6}\pi$	−·5	−·866	−40·9°
$\frac{4}{3}\pi$	−·866	−·5	−26·6°
$\frac{3}{2}\pi$	−1·	0	0
$\frac{5}{3}\pi$	−·866	·5	26·6°
$\frac{11}{6}\pi$	−·5	·866	40·9°
2π	0	1·	45°

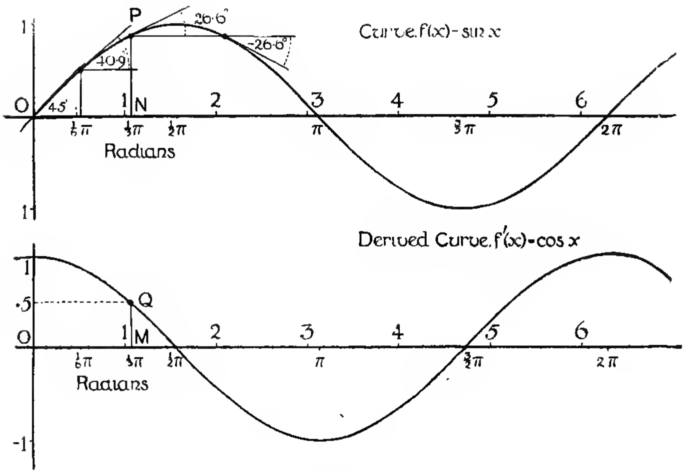


FIG. 79.

Unity (1 radian) on the horizontal scale and unity on the vertical scale must be represented by the same length.

Take any length, ON radians, and let NP be its sine. Take $OM = ON$ on the derived curve, and let MQ be its ordinate. Then MQ read on the scale of the derived curve ($=$ ·5 in figure) is the gradient at P on the original curve; and θ ($= 26{\cdot}6°$) is the inclination of the tangent at P, where $\tan\theta =$ ·5.

From $x = 0$ to $x = \frac{1}{2}\pi$, $f'(x)$ is positive and $f(x)$ rises. At $x = \frac{1}{2}\pi$, $f'(x) = 0$ and $f(x)$ is horizontal. From $x = \frac{1}{2}\pi$ to $x = \frac{3}{2}\pi$, $f'(x)$ is negative and $f(x)$ falls. The fall is steepest at $x = \pi$, when $\cos x$ is a minimum. The sine curve crosses the axis at an angle of 45°.

[By reference to the figure on p. 70, where the three functions $\sin x$, x, and $\tan x$ are represented together, it is to be noticed that their derived functions, viz. $\cos x$, 1, $\sec^2 x$, are each 1 when $x = 0$, and the three lines make the same angle (45°) at the start.]

The equation to the tangent to the curve $y = f(x)$ at the point $(x_1 y_1)$ is $\quad y - y_1 = (x - x_1) \times f'(x_1)$ (p. 136 (E)), where $f'(x_1)$ means the value obtained by writing the definite value x_1 in $f'(x)$.

Thus the tangent at $(x_1 y_1)$, a point on the parabola $y = x^2$, is

$$y - y_1 = (x - x_1) \times 2x_1, \quad \text{for} \quad f(x) = x^2,$$

$$f'(x) = 2x \text{ (Standard Form i)}, \quad f'(x_1) = 2x_1.$$

Since $x_1^2 = y_1$, this may be written $y + y_1 = 2xx_1$ (cf. p. 158).

Rules for Differentiation.

Differentiation of a Function of a Function.

If $y = F(u)$, where $u = f(x)$, then $D_x y = D_x u \times D_u y$.

For when x becomes $x + \delta x$, let u become $u + \delta u$, and y become $y + \delta y$.

$$\frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} \times \frac{\delta y}{\delta u} = \frac{(u + \delta u) - u}{\delta x} \times \frac{(y + \delta y) - y}{\delta u}$$

$$= \frac{f(x + \delta x) - f(x)}{\delta x} \times \frac{F(u + \delta u) - F(u)}{\delta u},$$

while δx, δy, δu are still finite.

When δx approaches zero, so do δu and δy.

Proceeding to the limit,

$$\underset{\delta x \to 0}{\mathrm{L}^t} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\mathrm{L}^t} \frac{f(x + \delta x) - f(x)}{\delta x} \times \underset{\delta u \to 0}{\mathrm{L}^t} \frac{F(u + \delta u) - F(u)}{\delta u}.$$

$$\therefore \quad \mathbf{D_x y = D_x u \times D_u y}. \quad \ldots \ldots \quad \text{(I)}$$

EXAMPLES. Let $y = \log_e x^2 = \log_e u$. $D_x u = D_x(x^2) = 2x$;

$$D_u y = D_u(\log_e u) = \frac{1}{u}.$$

$$\therefore\ D_x y = 2x \times \frac{1}{u} = 2x \times \frac{1}{x^2} = \frac{2}{x}.$$

Let $y = \sin(\log_e x) = \sin u$. $D_x u = D_x \log_e x = \frac{1}{x}$;

$$D_u y = D_u \sin u = \cos u.$$

$$\therefore\ D_x y = \frac{1}{x} \times \cos u = \frac{1}{x} \cdot \cos(\log_e x).$$

Particular Cases. Effect of Constants.

When $y = F(u) = u + k$, where k is constant,

$$D_u y = \mathrm{Lt}\, \frac{F(u+\delta u) - F(u)}{\delta u} = \mathrm{Lt}\, \frac{(u+\delta u + k) - (u+k)}{\delta u} = 1.$$

$$\therefore\ D_x y = D_x u,$$

and $\quad \mathbf{D_x\{f(x)+k\} = D_x\{f(x)\} = f'(x)}. \quad . \quad . \quad . \quad$ (II)

This is otherwise obvious from the consideration that the curves $y = f(x) + k$ and $y = f(x)$ differ only in their position relative to the axis of x and are parallel throughout.

When $y = F(u) = ku$, where k is constant,

$$D_u y = \mathrm{Lt}\, \frac{k(u+\delta u) - ku}{\delta u} = k.$$

$$\therefore\ D_x y = D_x u \times k,$$

and $\quad \mathbf{D_x\{k \cdot f(x)\} = kf'(x)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad$ (III)

EXAMPLE. Let $y = 6 \tan x$; then $D_x y = 6 \sec^2 x$.

When $u = f(x) = kx$, $D_x u = k$. $D_x y = k\, D_u y$.

$\quad \mathbf{D_x F(kx) = k \cdot D_{kx} F(kx)}. \quad . \quad . \quad . \quad . \quad . \quad$ (IV)

EXAMPLES. Let $y = \sin x^0 = \sin \frac{1}{180}\pi x$ radians.

$$D_x y = \tfrac{1}{180}\pi \cos x^0$$

Let $y = e^{kx}$. $\qquad D_x y = ke^{kx}$.

When $u = f(x) = x + k$, $D_x u = 1$, *and* $D_x y = D_u y$.

$\therefore\ \mathbf{D_x F(x+k) = D_{x+k} F(x+k)}. \quad . \quad . \quad . \quad . \quad$ (V)

EXAMPLE. Let $y = \sin(x+3)$; then $D_x y = \cos(y+3)$.

A little consideration will show that these results can be combined.

EXAMPLE. Let $y = 4\sin(ax+b)$; then $D_x y = 4a\cos(ax+b)$.

While Rule II relates to the shifting relative to the axis of x, Rule V similarly deals with parallel curves equal in all respects, but differing in their position relative to the axis of y.

Rules III and IV deal with curves related as an ellipse is related to a circle (p. 167).

Thus (Rule III) $y\,(=OQ) = x^2$, $y\,(=OP) = \frac{1}{2}x^2$.

If when $x = ON$, $y = NQ$, NP in the two curves, $NP = \frac{1}{2}NQ$. The rule gives that the gradient in the curve OP is half that in the curve OQ.

Again (Rule IV), let $y = (2x)^2$ be the curve OR.

If when

$y = OM$, $x = MQ$, MR

in the two curves,

$(2MR)^2 = OM = (MQ)^2$,

and $MR = \frac{1}{2}MQ$.

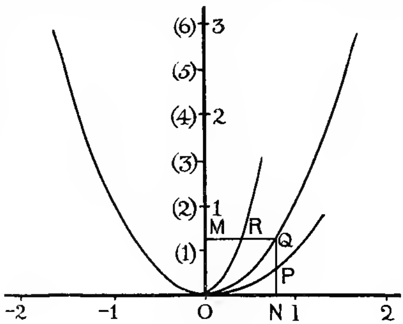


FIG. 80.

The rule gives that the gradient of the curve OR at R is twice that of OQ at Q, where the abscissa of Q is twice the abscissa of R.

If a curve, as $y = x^2$, is first drawn with units of abscissae and ordinates equal, as OQ, and then with the vertical scale halved (shown as (1), (2), (3) ... on the scale), we obtain such lines as OQ, OP representing the *same* equation. Halving the vertical scale clearly flattens the curve, and the two representations are related as curves under Rule III.

Conversely, curves $y = f(x)$, $y = kf(x)$ can be represented by the same line if different scales for ordinates are used, and curves $y = f(x)$, $y = f(kx)$ by the same line if different scales for abscissae are used.

Differentiation of Sums, Differences, Products, and Quotients.

Sum or Difference.

If $y = F(x) = f_1(x) \pm f_2(x) \pm f_3(x) \pm \ldots$, where $f_1, f_2, f_3, \ldots$ are any functions, then

$$F'(x) = D_x y = \underset{}{\text{Lt}} \frac{F(x+\delta x) - F(x)}{\delta x}$$

$$= \text{Lt} \frac{f_1(x+\delta x) \pm f_2(x+\delta x) \pm \ldots - \{f_1(x) \pm f_2(x) \pm \ldots\}}{\delta x}$$

$$= \mathbf{f_1'(x) \pm f_2'(x) \pm f_3'(x) \pm \ldots} \text{ (p. 105)} \quad . \quad . \quad . \quad \text{(VI)}$$

EXAMPLE. If $y = a + bx - cx^2 + dx^3$, $D_x y = b - 2cx + 3dx^2$.

Product.

If $y = F(x) = f_1(x) \times f_2(x)$, then

$$F'(x) = \text{Lt} \frac{f_1(x+\delta x) \times f_2(x+\delta x) - f_1(x) \times f_2(x)}{\delta x}$$

$$= \text{Lt} \frac{\{f_1(x+\delta x) - f_1(x)\} f_2(x+\delta x) + \{f_2(x+\delta x) - f_2(x)\} f_1(x)}{\delta x},$$

identically,

$$= \text{Lt} \frac{f_1(x+\delta x) - f_1(x)}{\delta x} \times \text{Lt} f_2(x+\delta x)$$

$$+ \text{Lt} \frac{f_2(x+\delta x) - f_2(x)}{\delta x} \times f_1(x),$$

$$\mathbf{D_x \{f_1(x) \times f_2(x)\} = f_1'(x) \cdot f_2(x) + f_2'(x) \cdot f_1(x).} \quad . \quad \text{(VII)}$$

EXAMPLES. If $y = x^n . \sin x$, $D_x y = nx^{n-1} . \sin x + \cos x . x^n$.
If $y = \sin x . \cos x$, $D_x y = \cos x . \cos x - \sin x . \sin x = \cos 2x$.
If $y = ax . \sin 2x$, $D_x y = a \sin 2x + 2ax \cos 2x$.

Quotient.

If $y = F(x) = f_1(x) \div f_2(x)$, then

$$F'(x) = \text{Lt} \left\{\frac{f_1(x+\delta x)}{f_2(x+\delta x)} - \frac{f_1(x)}{f_2(x)}\right\} \div \delta x$$

$$= \text{Lt} \frac{f_2(x) \{f_1(x+\delta x) - f_1(x)\} - f_1(x) \{f_2(x+\delta x) - f_2(x)\}}{\delta x . f_2(x+\delta x) . f_2(x)},$$

identically,

$$= \left\{ \mathop{L}^{t} \frac{f_1(x+\delta x) - f_1(x)}{\delta x} f_2(x) - \mathop{L}^{t} \frac{f_2(x+\delta x) - f_2(x)}{\delta x} \cdot f_1(x) \right\}$$

$$\times \mathop{L}^{t} \frac{1}{f_2(x+\delta x)} \cdot \frac{1}{f_2(x)}.$$

$$\mathbf{D_x} \frac{\mathbf{f_1(x)}}{\mathbf{f_2(x)}} = \frac{\mathbf{f_1'(x) \,.\, f_2(x) - f_2'(x) \,.\, f_1(x)}}{\{\mathbf{f_2(x)}\}^2} \quad . \quad . \qquad \text{(VIII)}$$

EXAMPLES. If $y = \tan x = \dfrac{\sin x}{\cos x}$,

$$D_x y = \frac{\cos x \,.\, \cos x - (-\sin x) \sin x}{(\cos x)^2} = \sec^2 x.$$

Obtain VIII from VII, treating $\dfrac{1}{f_2(x)}$ as a factor, using Rule I and Standard Form i, with $F(u) = \dfrac{1}{u} = u^{-1}$.

Inverse Functions.

If $y = f(x)$, it is often possible to express x as a function of y, say $x = F(y)$.

Then if the increments δx and δy are made to any pair of values of x, y,

$$\frac{\delta x}{\delta y} \times \frac{\delta y}{\delta x} = 1, \text{ always.}$$

$$\therefore \quad \mathop{L}^{t} \frac{\delta x}{\delta y} \times \frac{\delta y}{\delta x} = 1, \text{ and } \mathop{L}^{t} \frac{\delta x}{\delta y} \times \mathop{L}^{t} \frac{\delta y}{\delta x} = 1,$$

the limits being taken when δx, δy tend together to zero.

$$\therefore \quad \mathbf{D_y x \times D_x y = 1.} \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(IX)}$$

EXAMPLE. If $y = \sin^{-1} x$, $x = \sin y$.

$$\therefore \quad D_y(\sin y) \times D_x(\sin^{-1} x) = 1.$$

But $D_y(\sin y) = \cos y$ (Standard Form vii).

$$\therefore \quad D_x \sin^{-1} x = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}, \text{ as already found.}$$

Similarly, forms iv, v, xi, and xii can be obtained from ii, iii, viii, and ix.

The twelve standard forms and nine rules now given are sufficient for the differentiation of all the explicit functions, which have been so far used in this book, which (together with the hyperbolic functions, p. 247) are those generally included in elementary mathematics. Complete facility in using them can readily be obtained by practice. The examples in the following set may suffice.

NOTE. In the sequel $\log x$ stands for $\log_e x$.

EXAMPLES.

I. Find from the definition the derived functions in the following cases; draw the graphs of the functions, and compare the gradients at various points with the values of the derived functions:

(i) $\cos 2x$; (ii) $\tan 2x$; (iii) $2\sin x + 3\cos x$; (iv) $2x^2 - 3x + 4$; (v) $3x^3 - 4x$; (vi) $\frac{1}{x}$; (vii) $\frac{2}{x+3}$; (viii) $2\sqrt{1-\frac{1}{9}x^2}$. [Expand by the Binominal Series in powers of h.]

II. Differentiate the following functions, using the standard forms and rules. [The forms are referred to thus: iii, and the rules thus: IV.]

	Results:
1. $4x^4 - 5x^2$; i, III, VI.	$(16x^3 - 10x)$.
2. $\frac{4}{x^4}$; i.	$(-16x^{-5})$.
3. $(ax+b)^n$; i, I.	$\{na(ax+b)^{n-1}\}$.
4. 10^x; iii.	$(10^x \times 2{\cdot}3026\ldots)$
5. $\log(ax+b)$; iv, I, II, III.	$\left(\frac{a}{ax+b}\right)$.
6. $\tan\frac{x}{2}$.	$\left(\frac{1}{2}\sec^2\frac{x}{2}\right)$.
7. $(\sin x)^n$; i, vii, I.	$(n\cos x\sin^{n-1}x)$.
8. $\cos^2 3x$; i, viii, I, IV.	$(-3\sin 6x)$.
9. $\tan 2x - \tan x$; VI.	$(2\sec^2 2x - \sec^2 x)$.
10. $2\log(\tan x)$.	$(4\operatorname{cosec} 2x)$.
11. $\sqrt{a+x}$.	$\left(\frac{1}{2(a+x)^{\frac{1}{2}}}\right)$.
12. $\sqrt{a^2-x^2}$.	$\left(\frac{-x}{(a^2-x^2)^{\frac{1}{2}}}\right)$.
13. $\frac{1}{2x-5}$.	$\{-2(2x-5)^{-2}.\}$

14. $\sin x \,.\, \cos 2x$; VII. $\{\cos x\,(1 - 6\sin^2 x)\}$.

15. $\dfrac{x^2+2}{x+3}$; VIII. $\left(\dfrac{x^2+6x-2}{(x+3)^2}\right)$.

16. $\log(x + \sqrt{x^2 - a^2})$. $\left(\dfrac{1}{\sqrt{x^2-a^2}}\right)$.

17. $\sin 2x + 2x$. $(4\cos^2 x)$.

18. $\tan^{-1}\dfrac{x}{a}$. $\left(\dfrac{a}{a^2+x^2}\right)$.

19. $\log(\cos x)$. $(-\tan x)$.

20. $\sin x - x\cos x$. $(x\sin x)$.

21. $x(\log x - 1)$. $(\log x)$.

22. $\cot x$. $(-\operatorname{cosec}^2 x)$.

23. $\sec x$. $(\sec x \tan x)$.

24. $\operatorname{cosec} x$. $(-\operatorname{cosec} x \cot x)$.

25. $x\sqrt{x^2+a^2}$. $\left(\dfrac{2x^2+a^2}{\sqrt{x^2+a^2}}\right)$.

26. $\cot^{-1} x$. $\left(\dfrac{-1}{1+x^2}\right)$.

27. $\log(\tan \frac{1}{2}\theta)$. $\left(\dfrac{1}{\sin\theta}\right)$.

28. $x\sqrt{x^2+a^2} - a^2 . \log\left(\tan\left(\frac{1}{2}\cot^{-1}\dfrac{x}{a}\right)\right)$. $(2\sqrt{x^2+a^2})$.

29. $(\sin^2 x + 2)\cos x$. $(-3\sin^3 x)$.

30. $\tan x - x$. $(\tan^2 x)$.

31. $\frac{1}{2}\tan^2 x + \log\cos x$. $\tan^3 x$.

32. $\frac{1}{3}\tan^3 x - \tan x + x$. $\tan^4 x$.

33. $a\sin^{-1}\dfrac{x}{a} - \sqrt{a^2-x^2}$. $\left(\sqrt{\dfrac{a+x}{a-x}}\right)$.

34. $\tan^{-1}\sqrt{\dfrac{2x-a}{a}}$. $\left(\frac{1}{2}a \,.\, \dfrac{1}{x\sqrt{2ax-a^2}}\right)$.

35. $x\tan x - \dfrac{x^2}{2} + \log(\cos x)$. $(x \,.\, \tan^2 x)$.

36. $\cot 2x + \frac{1}{3}\cot^3 2x$. $(-2\operatorname{cosec}^4 2x)$.

37. $e^{-x}(\cos^2 x - \sin 2x + 2)$. $(-5e^{-x}\cos^2 x)$.

38. $e^x(x^3 - 3x^2 + 6x - 6)$. $(e^x \,.\, x^3)$.

Implicit Functions.

If y and x are connected by an equation of the form $y = f(x)$, where $f(x)$ does not involve y, then y is an *explicit* function of x. But if x and y are connected by an equation of the form $f(x, y) = 0$, where $f(x, y)$ depends on both x and y, then y is an *implicit* function of x. In both cases the equation represents a locus where (x, y) are the co-ordinates of a point. In the first, when x is known, there is only one possible value of y (as in the case of the parabola $y = x^2$), if f is rational in form; in the second case there may be more than one (as in the hyperbola $x^2 - y^2 - 1 = 0$).

We deal here only with implicit functions in which, when an increment of x is diminished to zero, the corresponding increment of y also diminishes to zero; thus in Figure 78 we assume that as N approaches M, $P'K$ tends to zero.

We also assume that y is capable of expression as an explicit function of x, though the function is not actually worked out, and that therefore $D_x y$ has a meaning.

Let $u = f(x, y) = 0$, and let the pairs of values (x, y), $(x + \delta x, y + \delta y)$ satisfy the equation. Then δu, the change in $f(x, y)$ due to the increments δx, δy, is zero, since u is always zero.

For example, if $f(x, y) = Ax^2 + By^2 - 1 = 0$,

$$0 = \delta u = \delta(Ax^2 + By^2 - 1) = \delta(Ax^2) + \delta(By^2).$$

Divide by δx and proceed to the limit when $\delta x = 0$.

$$0 = \text{L}^t \frac{\delta u}{\delta x} = D_x(Ax^2) + D_x(By^2) = D_x(Ax^2) + D_x y \, . \, D_y(By^2),$$

by Rule I,

$$= 2Ax + 2By \, . \, D_x y.$$

$$\therefore \quad D_x y = -\frac{Ax}{By}.$$

Then, as on p. 195, $y - y_1 = (x - x_1) D_x y$ touches $f(x, y) = 0$ at $(x_1 \, y_1)$.

In the case just taken, $y - y_1 = (x - x_1)\left(-\frac{Ax_1}{By_1}\right)$.

Equation of the tangent at x_1, y_1 is $Axx_1 + Byy_1 = Ax_1^2 + By_1^2$, i. e. $Axx_1 + Byy_1 = 1$, since $(x_1 \, y_1)$ is on the curve. (Compare p. 158.)

Again, in the equation $xy - c^2 = 0$,

$$0 = D_x(xy - c^2) = yD_x x + xD_x y = y + xD_x y.$$

$$D_x y = -\frac{y}{x}.$$

The tangent is $$y - y_1 = -\frac{y_1}{x_1}(x - x_1)$$

$$xy_1 + yx_1 = 2x_1 y_1 = 2c^2.$$

In the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$,

$$2ax + 2h(xD_x y + y) + 2byD_x y + 2g + 2fD_x y = 0.$$

The tangent is $(y - y_1)(hx_1 + by_1 + f) + (x - x_1)(ax_1 + hy_1 + g) = 0$, which is easily shown to be equivalent to

$$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c$$
$$= ax_1^2 + 2hx_1 y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0.$$ (p. 186, Ex. 3.)

Examples.

1. If xy^n is constant, $D_x y = -\dfrac{y}{nx}$.

2. If $x^3 - 3axy + y^3 - b^3 = 0$, $D_x y = (ay - x^2)/(y^2 - ax)$.

3. If $(x - y)^2(x + y)^3 - a^5 = 0$, $D_x y = (5x - y)/(5y - x)$.

4. If $\cos x \cos y - \cos 2x = \cos \alpha$, $D_x y = \dfrac{\tan x(4\cos x - \cos y)}{\sin y}$.

Maxima and Minima. Second Derived Function.

From the definition of a limit, and of a differential coefficient, $\dfrac{\delta y}{\delta x} = f'(x) + \epsilon$, where ϵ can be made as small as we please by diminishing δx. Take δx to be positive, as usual.

$$\therefore \quad \delta y = f'(x) . \delta x + \epsilon . \delta x.$$

If $f'(x)$ is positive, δy is positive, since $\epsilon . \delta x$ can be taken so small as not to affect the sign of the right-hand side of the equation.

Similarly, if $f'(x)$ is negative, δy is negative.

Thus as x increases, $f(x)$ increases if $f'(x)$ is positive, and decreases if $f'(x)$ is negative.

If $f'(x)$ is zero, δy is zero, and $f(x)$ is stationary.

[In Figure 81 δx is PK, δy is KP', $f'(x) = \tan\phi = \tan QPK$.

$$P'K = KQ + QP' = PK\tan\phi + QP'.$$

$$\delta y = \delta x \,.\, \tan\phi + QP'.$$

QP' is therefore $\epsilon \,.\, \delta x$.]

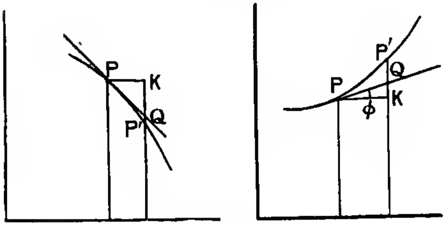


FIG. 81.

Let $f''(x)$ be the derived function of $f'(x)$, i.e. let

$$f''(x) = \underset{\delta x \to 0}{\mathrm{L}t} \frac{f'(x+\delta x) - f'(x)}{\delta x}.$$

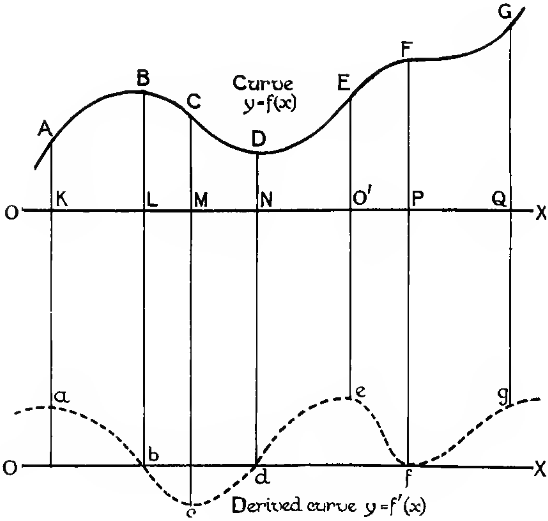


FIG. 82.

$f'(x)$, $f''(x)$ are called the first and second derived functions of $f(x)$.

By similar reasoning, as x increases, $f'(x)$ increases when $f''(x)$ is positive and decreases when $f''(x)$ is negative.

[In Figure 82 $f'(x)$ is positive and $f(x)$ increases from K to L, N to P, and P to Q; $f'(x)$ is negative and $f(x)$ decreases from L to N; $f'(x)$ is zero and $f(x)$ stationary at L, N, and P.

Again, from K to M $f'(x)$ decreases, and $f''(x)$ must be negative. $f(x)$ becomes less steep positively or steeper negatively. ABC is concave. From M to O', $f'(x)$ increases, $f''(x)$ must be positive, CDE is convex.

Compare the graph of $\sin x$ and its derived function on p. 194.]

If $f''(x)$ is negative between x_1 and x_2 ($x_2 > x_1$), then $f'(x)$ decreases between x_1 and x_2, and if at some value x', between x_1 and x_2, $f'(x')$ is zero, it follows that $f'(x')$ is positive from x_1 to x' and negative from x' to x_2; therefore $f(x)$ increases from x_1 to x', is stationary at x', and decreases from x' to x_2. Hence if $f''(x')$ is negative and $f'(x')$ zero, $f(x')$ is greater than the values on either side of it. In other words $f(x')$ is a *maximum* value of $f(x)$.

[In Figure 82 OK, OL, OM may be taken to be x_1, x', x_2].

On the other hand, if $f''(x')$ is positive when $f'(x')$ is zero, $f(x')$ is less than the values of $f(x)$ on either side of it, and $f(x')$ is a *minimum* value of $f(x)$. [OM, ON, OO'.]

If, when $f'(x')$ is zero, $f''(x')$ is also zero, no conclusion can be drawn as to the existence of a maximum or minimum value without using a higher derived function. [P.]

The argument is independent of and does not postulate the possibility of graphic representation, but its nature is most easily comprehended from Figure 82.

Hence, to find the maxima and minima of a function of one variable, solve the equation $f'(x) = 0$. If it is not evident *a priori* whether $f(x_1)$, x_1 being a root, is a maximum or minimum, test the sign of $f''(x_1)$.

It is nearly self evident that, as x changes, maxima and minima succeed each other alternately.

EXAMPLES.

1. Required the maximum product of two quantities whose sum is k.

Let x and $k-x$ be the quantities

$$f(x) = x(k-x) = kx - x^2,$$
$$0 = f'(x) = k - 2x,$$
$$x' = \tfrac{1}{2}k,\ f(x') = \tfrac{1}{4}k^2,$$
$$f''(x) = -2, \text{ negative.}$$

Hence $\frac{1}{4}k^2$ is a maximum.

Here the second differentiation is unnecessary, as it is evident that a maximum exists and no minimum except $-\infty$.

2. Let $f(x) = 3x^3 - 5x^2 - 11x - 10$. (Figure, Section IV, p. 94.)

$$0 = f'(x) = 9x^2 - 10x - 11. \quad x = \tfrac{1}{9}(5 \pm \sqrt{124}) = 1{\cdot}79 \text{ or } -{\cdot}68.$$
$$f''(x) = 18x - 10.$$

$\therefore$ $f''(1{\cdot}79)$ is positive, and $x = 1{\cdot}79$ gives a minimum,

$f''(-{\cdot}68)$ is negative, and $x = -{\cdot}68$ gives a maximum.

NOTE. $f''(x) = 0$ if $x = \frac{5}{9}$. Here $f'(x)$ is a maximum or minimum, and since $f'''(x) = 18$ and is positive, it is a minimum, i.e. the curve has greatest downward slope at $x = \frac{5}{9}$.

Find the maxima and minima of the following functions, verifying the results by rough graphs:

3. $(x-1)(x-2)(x-3)$. 4. $x^4 - 5x^2 + 7x - 8$.

5. $2\sin x + 3\cos x$. 6. $x\log_e x$. 7. $\dfrac{4x^2}{3x-2}$.

8. The volume of a rectangular solid, the sum of whose edges is given, is greatest when it is a cube. [Whatever length is taken for one edge, by Ex. 1 the volume is greatest when the other two are equal.]

9. The maximum volume of a rectangular solid, whose total surface is given, is a cube. [If A, B, C are the areas of the faces and V the volume, $V^2 = ABC$. Then use Ex. 8.]

INTEGRATION.

Integration is the process of deducing the original function when the derived function is known.

$f(x)$ and $f(x) + C$, where C is a constant, have the same derived function, viz. $f'(x)$.

Conversely, if two functions have the same derived function they only differ by a constant: for let $f'(x) = F'(x)$ for all values of x, x_1 be any assigned value of x, and $y_1 = f(x_1)$, $y_2 = F(x_1)$. Then, in the notation already used,

$$\delta y_1 = f(x_1 + \delta x_1) - f(x_1)$$
$$= f'(x_1) \, . \, \delta x_1 + \epsilon \, . \, \delta x_1$$

and $\quad \delta y_2 = F'(x_1) \, . \, \delta x_1 + \epsilon' . \, \delta x_1 .$

$$\therefore \; \delta y_2 = \delta y_1 + (\epsilon - \epsilon') \, \delta x_1,$$

since $f'(x_1) = F'(x_1)$.

$$\therefore \; (y_2 + \delta y_2) - (y_1 + \delta y_1)$$
$$= y_2 - y_1 + (\epsilon - \epsilon') \, \delta x_1 .$$

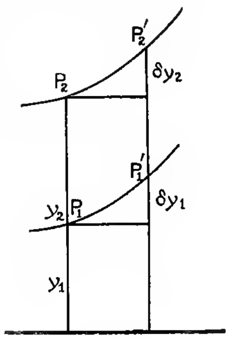


FIG. 83.

Hence as we make small finite increments in x, the differences between successive pairs of values of y are as nearly equal as we please. That is, $y_2 - y_1$, i.e. $F(x) - f(x)$, remains unchanged.

[If the functions are represented graphically, the curves are parallel to each other.]

The integral, or original, function is obtained by recognizing the given function as derived from a known function. There is no general direct way of finding it.

Thus, given $f'(x) = \cos x$, we recognize from Standard Form VI that $f(x) = \sin x$, but we must write this $\sin x + C$ for the reasons just given.

The following table, based on the Standard Forms, is easily verified, by obtaining the first column from the second.

FUNCTION.	INTEGRAL.
1. $a(bx+c)^n$.	$\dfrac{a(bx+c)^{n+1}}{(n+1) \, . \, b} + C.$
2. a^x.	$a^x \log_a e + C.$
3. $\dfrac{a}{x+b}$.	$a \log_e (x+b) + C.$
4. $a \sin (bx+c)$	$-\dfrac{a}{b} \cos (bx+c) + C.$
5. $a \cos (bx+c)$.	$\dfrac{a}{b} \sin (bx+c) + C.$

6. $\sec^2(bx+c)$. $\quad \frac{1}{b}\tan(bx+c)+C.$

7. $\frac{1}{\sqrt{a^2-x^2}}$. $\quad \sin^{-1}\frac{x}{a}+C.$

8. $\frac{1}{a^2+x^2}$. $\quad \frac{1}{a}\tan^{-1}\frac{x}{a}+C.$

The forms just given are those for which the integral function is simplest. If we start at the other end of the problem and try to find the integral function of the simplest derived functions, we have to devise methods whose justification is their success. Such methods have been found for very many, but not all, simple functions.

A special notation is in use for the problem of integration, which originated from the relation between integration and summation shown on p. 213. $\int$, a form of the letter *s*, is called the sign of integration. If $\phi(x)$ is written for $f'(x)$, then $f(x)$ is the integral function for $\phi(x)$. This is written

$$\int \phi(x)\,.\,dx = f(x)+C.$$

The insertion of dx is explained on pp. 213–14.

This is not a proved equation, but only a convenient way of putting the general statement, '$\phi(x)$ is the derived function of $f(x)$' when $\phi(x)$ is known and $f(x)$ is to be found.

Thus, $\int x^n dx = \frac{x^{n+1}}{n+1}+C.$

The following are important integrals. The reader should confine himself to assenting to the correctness of the steps taken, without considering *why* the particular path is chosen. The processes can be verified by working them backwards.

$$9.\ \int\frac{dx}{a^2-x^2}=\int\frac{1}{2a}\left(\frac{1}{a-x}+\frac{1}{a+x}\right)dx=\frac{1}{2a}\left\{\int\frac{dx}{a-x}+\int\frac{dx}{a+x}\right\}$$

$$=\frac{1}{2a}\{-\log(a-x)+\log(a+x)\}+C=\frac{1}{2a}\log\frac{a+x}{a-x}+C.$$

[Here we have used the fact that the integral of a sum is the sum of the integrals of its terms, the converse of Rule VI, p. 198.]

10. $\int \frac{dx}{a+bx+cx^2}$.

Case (i) when $b^2 > 4ac$. Let $a+bx+cx^2 = c(x-\alpha)(x-\beta)$. α and β are real.

$$\int \frac{dx}{c(x-\alpha)(x-\beta)} = \frac{1}{c}\int \frac{1}{\alpha-\beta}\left(\frac{1}{x-\alpha} - \frac{1}{x-\beta}\right)dx$$

$$= \frac{1}{(\alpha-\beta)c}\left(\int \frac{dx}{x-\alpha} - \int \frac{dx}{x-\beta}\right) = \frac{1}{c(\alpha-\beta)}\log\frac{x-\alpha}{x-\beta} + C$$

$$= \frac{1}{\sqrt{b^2-4ac}}\log\frac{x-\alpha}{x-\beta} + C = \frac{1}{k}\log\frac{2cx+b-k}{2cx+b+k} + C,$$

where $k^2 = b^2-4ac$.

Case (ii) when $b^2 < 4ac$.

$$\int \frac{dx}{a+bx+cx^2} = \int \frac{dx}{c\left(x+\frac{b}{2c}\right)^2 + \frac{4ac-b^2}{4c}} = \frac{1}{c}\int \frac{dx'}{x'^2+d^2},$$

where $x' = x + \frac{b}{2c}$, and $\therefore$ $\delta x' = \delta x$, and $d = \frac{\sqrt{4ac-b^2}}{2c}$,

$$= \frac{1}{cd}\tan^{-1}\frac{x'}{d} + C = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2cx+b}{\sqrt{4ac-b^2}} + C.$$

Case (iii) when $b^2 = 4ac$.

$$\int \frac{dx}{c\left(x+\frac{b}{2c}\right)^2} = -\frac{1}{c\left(x+\frac{b}{2c}\right)} = -\frac{2}{2cx+b}.$$

11. $\int \tan x \,.\, dx$. Let $x' = \cos x$. Then $\delta x' = -\sin x \,.\, \delta x$.

$$\int \tan x \,.\, dx = \int -\frac{dx'}{x'} = -\log x' + C = -\log(\cos x) + C.$$

12. $\int \frac{dx}{\sin x} = \int \frac{\sec^2 \frac{1}{2}x}{2\tan\frac{1}{2}x}\,.\,dx = \int \frac{dx'}{x'}$, where $x' = \tan\frac{1}{2}x$.

$$= \log(\tan\tfrac{1}{2}x) + C.$$

13. $\int \cos^2 x \,.\, dx = \int \frac{1}{2}(\cos 2x + 1)\,.\,dx = \frac{1}{4}\sin 2x + \frac{1}{2}x + C.$

14. $\int \sqrt{a^2-x^2}\,.\,dx.$

Let $x = a\sin\theta$, $\delta x = a\cos\theta\,.\,\delta\theta$.

$$\int \sqrt{a^2-x^2}\,.\,dx = a^2\int \cos^2\theta\,.\,d\theta = a^2\left(\tfrac{1}{4}\sin 2\theta+\tfrac{1}{2}\theta+C\right)$$

$$= \tfrac{1}{2}x\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C.$$

15. $\displaystyle\int \frac{dx}{\sqrt{a^2+x^2}}\cdot$ Let $x = a\cot\theta.$

Expression $\displaystyle = -\int \frac{\operatorname{cosec}^2\theta\,.\,d\theta}{\operatorname{cosec}\theta} = -\log\left(\tan\frac{\theta}{2}\right)+C$ from Ex. 12

$$= \log\cot\left(\tfrac{1}{2}\cot^{-1}\frac{x}{a}\right)+C.$$

16. $\int \sin ax\,.\cos bx\,.\,dx = \tfrac{1}{2}\int\{\sin(a+b)\,x+\sin(a-b)x\}\,dx$

$$= -\frac{1}{2(a+b)}\cos(a+b)\,x-\frac{1}{2(a-b)}\cos(a-b)\,x+C.$$

17. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \log_e\left(x+\sqrt{x^2-a^2}\right)+C,$ as can be verified by differentiation. (See p. 201, Ex. 16.)

EXAMPLES.

1. $\int(4x^3+3x^2+2x)\,dx.$ $\quad(x^4+x^3+x^2+C).$

2. $\displaystyle\int \frac{x+2}{x+1}\,dx.$ $\quad\{x+\log(x+1)+C\}.$

3. $\displaystyle\int \frac{xdx}{x^2+a^2}\cdot$ [Write $x^2 = u.$] $\quad\{\tfrac{1}{2}\log(x^2+a^2)+C\}.$

4. $\int 10^{4x}\,dx.$ $\quad(\tfrac{1}{4}\cdot 10^{4x}\log_{10}e+C).$

5. $\int \sin 3x\,dx.$ $\quad(-\tfrac{1}{3}\cos 3x+C).$

6. $\int \sin^2 x\,dx.$ $\quad\{\tfrac{1}{2}(x-\sin x\cos x)+C\}.$

7. $\int \sin^3 x\,dx.$ [Use formula $\sin 3x = 3\sin x-4\sin^3 x.$]

$(-\tfrac{3}{4}\cos x+\tfrac{1}{12}\cos 3x+C).$

8. $\int \cos^3 x\,dx.$ $\quad(\tfrac{3}{4}\sin x+\tfrac{1}{12}\sin 3x+C).$

9. $\int \sec x\,.\,dx.$ $\left[\text{Write }\tan\dfrac{x}{2}=t.\right]$ $\quad\{\log\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}x)+C\}.$

10. $\int \sin x\,.\,\sin 2x\,dx.$ $\quad(-\tfrac{1}{6}\sin 3x+\tfrac{1}{2}\sin x+C).$

11. $\int \cot x\, dx$. [Write $\sin x = x'$.] $\{\log(\sin x) + C\}$.

12. $\int \frac{x^2 dx}{x^6 + a^6}$. [Write $x^3 = u$.] $\left(\frac{1}{3a^3} \tan^{-1}\frac{x^3}{a^3} + C\right)$.

13. $\int \frac{x\, dx}{\sqrt{x^2 + a^2}}$. $\left[\begin{array}{c}\text{Write } x = a \tan\theta \\ \text{or } x^2 = u.\end{array}\right]$ $(\sqrt{x^2 + a^2} + C)$.

14. $\int \frac{dx}{x^2 + 2x}$. $\left(\frac{1}{2}\log\frac{x}{x+2}\right)$.

15. $\int \frac{dx}{x^2 + 2x + 1}$. $\left(-\frac{1}{x+1} + C\right)$.

16. $\int \frac{dx}{x^2 + 2x + 2}$. $\{\tan^{-1}(x+1) + C\}$.

17. $\int \frac{x\, dx}{x^4 + 16}$. $(\frac{1}{8}\tan^{-1}\frac{1}{4}x^2 + C)$.

18. $\int \frac{2x\, dx}{x^4 - 16}$. $\left(\frac{1}{8}\log\frac{x^2 - 4}{x^2 + 4} + C\right)$.

19. $\int \frac{dx}{2 + 3x - 5x^2}$. $\left(\frac{1}{7}\log\frac{2 + 5x}{1 - x} + C\right)$.

20. $\int \frac{x}{x^2 - a^2} dx$. $\{\frac{1}{2}\log(x^2 - a^2) + C\}$.

21. $\int \frac{x\, dx}{x^2 - 3x + 2}$. $\left(\log\frac{(x-2)^2}{x-1} + C\right)$.

22. $\int (x+a)^{\frac{2}{7}} dx$. $\{\frac{7}{9}(x+a)^{\frac{9}{7}} + C\}$.

23. $\int \sec^4 x\, .\, dx$. [Write $\tan^2 x = t$.] $(\tan x + \frac{1}{3}\tan^3 x + C)$.

Integration by Parts.

If u and v are functions of x,

$$\mathbf{D_x(uv) = uD_x v + vD_x u.}$$

$\therefore\ uD_x v = D_x(uv) - vD_x u$, for all values of x; and therefore the quantities of which the left- and right-hand sides are derived functions differ only by a constant.

$$\therefore \int uD_x v\, .\, dx = \int \{D_x(uv) - vD_x u\}\, dx + \text{const.}$$
$$= \int D_x(uv)\, .\, dx - \int vD_x u\, .\, dx + \text{const.} = uv + C - \int vD_x u\, .\, dx.$$

This formula is of frequent use when we have to integrate a function, one of whose factors we recognize as a known derived function.

EXAMPLES.

1. $\int x \sin x \, dx$. Let $u = x$, $v = -\cos x$.

$$\int x . \sin x \, dx = x(-\cos x) - \int (-\cos x) . 1 \, dx.$$
$$= -x \cos x + \sin x + C.$$

2. $\int \log x . dx$. Let $u = \log x$, $v = x$.

$$\int \log x . dx = x \log x - \int x D_x \log x . dx = \log x - \int x . \frac{1}{x} dx$$
$$= x \log x - x + C.$$

3. $$\int \sqrt{x^2+a^2} \, dx = x\sqrt{x^2+a^2} - \int \frac{x^2}{\sqrt{x^2+a^2}} dx.$$

But $$-\int \frac{x^2}{\sqrt{x^2+a^2}} dx = -\int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}} dx$$
$$= -\int \sqrt{x^2+a^2} \, dx + a^2 \int \frac{dx}{\sqrt{x^2+a^2}}.$$

$$\therefore \; 2\int \sqrt{x^2+a^2} \, dx = x\sqrt{x^2+a^2} - a^2 \log \left\{\tan\left(\tfrac{1}{2}\cot^{-1}\frac{x}{a}\right)\right\}.$$

4. $\int \sin^3 x \, dx$. Let $u = \sin^2 x$, $v = -\cos x$.

$$\int \sin^3 x . dx = -\sin^2 x \cos x - \int (-\cos x)(2 \sin x \cos x) \, dx + \text{const.},$$
$$= -\sin^2 x \cos x + 2\int \sin x (1 - \sin^2 x) \, dx + \text{const.},$$
$$= -\sin^2 x \cos x + 2\int \sin x \, dx - 2\int \sin^3 dx + \text{const.}$$
$$\therefore \; 3\int \sin^3 x . dx = -\sin^2 x \cos x + 2\int \sin x . dx + \text{const.}$$
$$\therefore \; \int \sin^3 x . dx = -\tfrac{1}{3}\sin^2 x \cos x - \tfrac{2}{3}\cos x + \text{const.}$$

5. $$\int \sin^n x . dx = -\sin^{n-1} x \cos x$$
$$-\int (-\cos x)(\overline{n-1} \sin^{n-2} x \cos x \, dx),$$
$$= -\sin^{n-1} x . \cos x + (n-1)\int \sin^{n-2} x (1 - \sin^2 x) \, dx,$$
$$= -\sin^{n-1} x \cos x + (n-1)\left\{\int \sin^{n-2} x \, dx - \int \sin^n x \, dx\right\}.$$
$$\therefore \; \int \sin^n x \, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x . dx + C.$$

This process can be applied again and again, till the expression is completely integrated, when n is a positive integer.

6. $$\int \cos^n x . dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x . dx + C.$$

7. $$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx$$
$$= e^x \cos x + e^x \sin x - \int e^x \cos x \, dx.$$
$$\therefore \; \int e^x \cos x \, dx = \tfrac{1}{2} e^x (\sin x + \cos x) + C.$$

8. $$\int e^x \sin x \, dx = \tfrac{1}{2} e^x (\sin x - \cos x) + C.$$

9. $$\int e^{ax} \sin (bx + c) \, dx = \frac{e^{ax}}{a^2+b^2}\{a \sin (bx+c) - b \cos (bx+c)\} + C.$$

Definite Integrals. Integration and Summation.

Let $f(x)$ be such that it can be represented by a graph, CD.

Let $OA = a, \quad OB = b$,

then $AC = f(a)$, $BD = f(b)$.

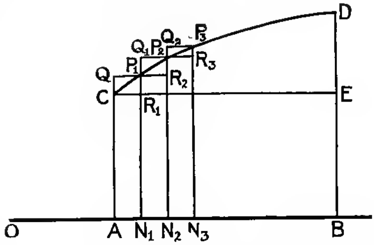


FIG. 84.

Let $\delta x = \dfrac{AB}{n}$, and cut off AN_1, $N_1N_2 \ldots N_{n-1}B$ each $= \delta x$. Then

$$b = a + n\,\delta x.$$

Let N_1P_1, N_2P_2, ... be ordinates of the curve.

Complete the two sets of rectangles AQP_1N_1, $N_1Q_1P_2N_2$,... and ACR_1N_1, $N_1P_1R_2N_2$, ... as in the figure.

First suppose that the ordinates increase as x increases.

Let S, S' be the area of the rectangular figures

$$ACR_1P_1R_2P_2 \ldots, \text{ and } AQP_1Q_1P_2Q_2 \ldots D.$$

$$S = \delta x\,[f(a) + f(a+\delta x) + f(a+2\delta x) + \ldots + f(a+\overline{n-1}\,\delta x)],$$

$$S' = \delta x\,[f(a+\delta x) + f(a+2\delta x) + \ldots + f(a+n\,\delta x)].$$

$$S' - S = \delta x\,\{f(a+n\delta x) - f(a)\} = \delta x\,\{f(b) - f(a)\}$$

$$= ED \times \delta x,$$

where CE is parallel to AB and meets BD at E,

$$= ED \times AB \div n.$$

Let $\phi(x)$ be such that $\phi'(x) = f(x)$ for all values of x from a to b; i.e. $\phi(x) = \int f(x)\,dx + C$.

Then $\phi(a+\delta x) - \phi(a) = \delta x\,\{\phi'(a) + \epsilon_1\}$ (p. 192, definition of limit), $\quad = \delta x\,\{f(a) + \epsilon_1\}$,

$$\phi(a+2\delta x) - \phi(a+\delta x) = \delta x\,\{f(a+\delta x) + \epsilon_2\}$$

. . . .

$$\phi(b) - \phi(a+\overline{n-1}\,\delta x) = \delta x\,\{f(a+\overline{n-1}\,\delta x) + \epsilon_{n-1}\}.$$

Adding,

$\therefore\ \phi(b) - \phi(a) = S + \delta x\,\{\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n\}$, where each of the ϵ's may be made less than any assigned quantity by diminishing δx. Let ϵ be the greatest of the ϵ's.

Then $\delta x\,(\epsilon_1+\epsilon_2 \ldots \epsilon_n) \not> n\,.\,\delta x \times \epsilon \not> AB \times \epsilon$, since $AB = n\delta x$,
$$\not> (b-a)\times\epsilon.$$

Hence, if δx is diminished by increasing n, the number of points of section, $\phi\,(b)-\phi\,(a)$ may be made to differ from S by as little as we please.

But $S'-S = \dfrac{ED \times AB}{n}$, which may be made as small as we please.

$\therefore$ $\phi\,(b)-\phi\,(a)$ is the limit both of S, S', and of any areas intermediate between them when n is increased indefinitely.

If we define the curve (compare p. 66) as the limit of the rectilinear figure $CP_1P_2 \ldots D$, then $\phi\,(b)-\phi\,(a)$ is the area included between AB and the curve CD.

If the ordinates first increase and then diminish, or vice versa, the argument that $\phi\,(b)-\phi\,(a)$ is the limit of S is unaffected—whereas in the case taken all the ϵ's were positive, now some would be negative—and it is easy to see from a figure that $S'-S$ tends more rapidly to zero.

The process is generally written $\displaystyle\int_a^b f(x)\,.\,dx = \Big[\phi\,(x)\Big]_a^b$. The left-hand side is called a *definite integral*, and a and b the *limits of integration.*

EXAMPLE. To find the area of part of the parabola $x^2 = 4py$.

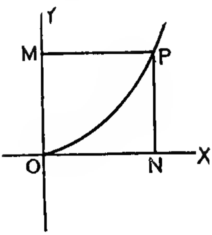


FIG. 85.

Let O be the vertex, PN, PM perpendiculars on the axes from a point on the curve. Let $ON = b$. Then $NP = \dfrac{b^2}{4p}$.

Let $$f(x) = \frac{x^2}{4p}.$$

Then $\phi\,(x) = \int f(x)\,dy + \text{const.} = \dfrac{x^3}{12p} + \text{const.}$

Hence the area $ONP = \phi\,(b)-\phi\,(0)$
$$= \left(\frac{b^3}{12p} + C\right) - (0+C) = \frac{b^3}{12p} = \tfrac{1}{3}\,.\,ON\,.\,NP,$$
and thence the area $OMP = \tfrac{2}{3}\,ON\,.\,NP.$

The constant of integration may evidently be omitted in the work.

The use of definite integrals is not confined to areas, as the following examples show.

To find the volume of a sphere.

Let OA be a semi-diameter and the axis of x. Let a be the radius.

Let $ON = x$, $NN' = \delta x$.

Take sections of the sphere, perpendicular to the axis of x, namely the circles on PQ, $P'Q'$ as diameters.

Draw PL perpendicular to $N'P'$, and $P'K$ to NP.

Suppose two cylinders constructed, each with axis NN', one with radius NP, the other with radius NK.

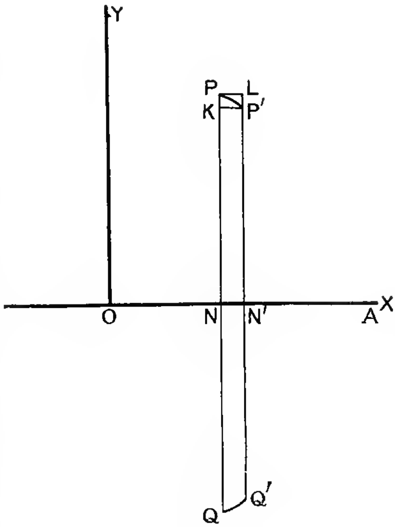


FIG. 86.

The volume of the zone of the sphere bounded by the circular sections is intermediate between those of the cylinders LPQ and $P'KQ'$, that is, intermediate between $\pi y^2 . \delta x$ and $\pi(y+\delta y)^2 . \delta x$, where $y = NP$, and $x^2 + y^2 = a^2$.

Suppose successive elements δx cut off from O to A and two cylinders constructed on each element. The hemisphere is intermediate in volume between the two sets of cylinders.

As on p. 214, it can be shown that in the limit, when δx is diminished, the three volumes, hemisphere, sum of outer and sum of inner cylinders are equal.

Hence the volume of the hemisphere is $\int_0^a \pi y^2 dx$, and of the sphere

$$2\int_0^a \pi(a^2 - x^2)\,dx = 2\pi\left[a^2x - \frac{x^3}{3}\right]_0^a = 2\pi\left(a^3 - \frac{a^3}{3}\right) = \tfrac{4}{3}\pi a^3.$$

EXAMPLE. The volume of a right circular cone is $\frac{1}{3}\pi r^2 h$, where r is the radius of the base and h the altitude.

[Take the axis of the cone as axis of x, and proceed as with the sphere.]

Examples.

1. Evaluate $\int_0^1 (x^2+3x)dx$, $\int_0^{\frac{\pi}{4}} \sin x\, dx$, $\int_0^1 10^x . dx$, $\int_1^a \frac{dx}{x}$.

2. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\left[\; 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,.\,dx. \right]$$

3. Find the area of the curve $y = e^x$ from $x = 0$ to $-\infty$. Deduce the area of $y = \log_e x$ and $y = \log_{10} x$ from $x = 0$ to 1. [1, 1, ·4343.]

4. Find the area of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from $x = a$ to $x = c$. $\left[\frac{b}{a} c\sqrt{c^2-a^2} - ab\log\frac{c+\sqrt{c^2-a^2}}{a}.\right]$

Differential Equations.

It is proposed to give a few examples illustrating the solution of equations involving the differential coefficient.

If a moving particle is at a distance y from a fixed point in its path at a time x, its velocity may be defined as

$$\mathop{\mathrm{Lt}}_{\delta x = 0} \frac{\delta y}{\delta x} = v, \text{ i.e. } v = D_x y = f'(x),$$

and its acceleration may be defined as $D_x v = f'(v) = f''(x)$.

Example 1. Constant acceleration a. $f'(v) = a$,

$$v = \int a dx + C = ax + C.$$

If $v = u$ when $x = 0$ (initial velocity),

$$u = a \times 0 + C.$$

$$\therefore\; v = ax + u$$

[or if v is the velocity accumulated from zero time to time t

$$v = \int_0^t a dx = \Big[ax\Big]_0^t = at].$$

Again, $D_x y = f'(x) = v = ax + u$,

$$y = \int (ax+u)\,dx = \tfrac{1}{2}ax^2 + ux + C.$$

If $y = 0$ when $x = 0$, this becomes $y = \frac{1}{2}ax^2 + ux$

[or if y is distance from zero time to time t,

$$y = \int_0^t (ax+u)\,dx = \Big[\tfrac{1}{2}ax^2 + ux\Big]_0^t = ut + \tfrac{1}{2}at^2].$$

EXAMPLE 2. Acceleration towards a fixed origin and proportional to its distance from it. [Simple harmonic motion.]

Then the velocity increases as y diminishes, and

$$D_x v = f''(x) = -ky, \ k \text{ positive.}$$

Multiply both sides by v. $v \,.\, D_x v = -ky D_x y$.

$$\therefore \int (v D_x v) \,.\, dx = -\int ky\,dy + \text{const.}$$

$$\therefore \tfrac{1}{2}v^2 = -\frac{k}{2}y^2 + C.$$

If $v = 0$ when $y = a$, $\quad 0 = -\dfrac{k}{2}a^2 + C.$

$$\therefore v^2 = k(a^2 - y^2).$$

$$\therefore D_x y = v = \sqrt{a^2 - y^2} \,.\, \sqrt{k}.$$

$$\therefore D_y x = \frac{1}{D_x y} = \frac{1}{\sqrt{k}\sqrt{a^2 - y^2}},$$

$$x = \int \frac{dy}{\sqrt{k}\sqrt{a^2 - y^2}} = \frac{1}{\sqrt{k}} \sin^{-1}\frac{y}{a} + C.$$

$$\therefore y = a \sin(\sqrt{k} \,.\, x + C).$$

EXAMPLE 3. Acceleration negative and varying as velocity. [Possible law of motion in a resisting medium.]

$$D_x v = -kv. \quad \therefore D_v x = \frac{1}{-kv}; \quad x = -\int \frac{dv}{kv} = -\frac{1}{k}\log v + C.$$

If $v = u$, when $x = 0$ (initial velocity), $C = \dfrac{1}{k}\log u$;

$$x = \frac{1}{k}\log\frac{u}{v}; \quad v = ue^{-xk};$$

$$y = \int ue^{-xk}\,dx = -\frac{u}{k}e^{-xk} + C = \frac{u}{k}(1 - e^{-xk}) \text{ if } y = 0 \text{ when } x = 0.$$

Notice that if x is increased indefinitely, the moving particle comes to rest after travelling a distance $\dfrac{u}{k}$.

EXAMPLES.

1. Find the velocity after time t, and the distance described in the time, of a particle projected vertically upwards with velocity u, the retardation due to gravity being taken as constant (g).

2. A body is projected with velocity u and is subject to an acceleration $g - kv^2$, where g and k are constant. Find the velocity which it tends to obtain when the time is indefinitely increased. [Terminal velocity.]

3. Solve the equations

(i) $D_x y = ax + b.$ (ii) $D_x y = ax^2 + bx + c.$

(iii) $D_x y = a \sin x.$ (iv) $D_x y = e^x \sin x.$

(v) $y D_x y = 2a.$ (vi) $(y+x)(1 + D_x y) = x^2.$
[Put $y + x = z$.]

(vii) $\sec^2 y \,.\, D_x y = \sin x.$ (viii) $(D_x y)^2 = \dfrac{a^2}{1-x^2}.$

(ix) $D_x y = y \tan x.$ (x) $\dfrac{1}{x} D_x\left(\dfrac{1}{x} D_x y\right) = a.$

Partial Differentiation.

Preliminary illustration. If y is the pressure of a given quantity of gas, u its absolute temperature, and v its density, then $y = k \,.\, uv$ for all variations of pressure, temperature, and density over a certain range, where k is a constant. If increments δu, δv, δy take place together,

$$y + \delta y = k(u + \delta u)(v + \delta v).$$

$$\therefore \; \delta y = k(v \delta u + u \delta v) + k \delta u \,.\, \delta v. \quad \ldots \quad \text{(i)}$$

Suppose that u, v, and therefore y are functions of a third quantity x (e.g. the time during which the gas is subject to experiment), then

$$\frac{\delta y}{\delta x} = k\left(v \frac{\delta u}{\delta x} + u \frac{\delta v}{\delta x}\right) + k \,.\, \frac{\delta u}{\delta x} \,.\, \delta v.$$

Now let δx tend to zero with the other increments, and we have

$$D_x y = kv \,.\, D_x u + ku \, D_x v. \quad \ldots \quad \text{(ii)}$$

Returning to equation (i) we see that [when, as in equation (ii), $\delta u \,.\, \delta v$ is neglected in comparison with δu or δv] δy is the sum of two parts, viz. $kv \,.\, \delta u$, which is the increase due to temperature and is the same as if the density had not changed, and $ku \,.\, \delta v$, which is the increase due to density, and is the same as if the temperature had not changed.

kv is the result of differentiating y with respect to u, in the case where v is constant.

ku is the result of differentiating y with respect to v, in the case where u is constant.

Such a differentiation is called partial. We may write it thus:

$$\underset{v\text{ const.}}{D_u y} = kv\,; \qquad \underset{u\text{ const.}}{D_v y} = ku.$$

Equation (ii) is then

$$D_x y = \underset{v\text{ const.}}{D_u y} \,.\, D_x u + \underset{u\text{ const.}}{D_v y} \,.\, D_x v. \quad . \quad . \quad . \quad \text{(iii)*}$$

Equation (ii) shows that in this case the whole rate of change $(D_x y)$ is the sum of the rates due to change of temperature and change of density taken separately.

Equation (iii) can be shown, as follows, to be true whatever function y is of u, v. The proof cannot be made easy, without inaccuracy.

Let $y = F(u, v)$, where u and v are both functions of some variable x. Let none of the differentials become infinite in the processes.

Let an increment δx result in increments δu, δv, δy.

Then $\delta y = F(u+\delta u,\ v+\delta v) - F(u, v)$

$$= \{F(u+\delta u,\ v+\delta v) - F(u,\ v+\delta v)\}$$
$$+ \{F(u,\ v+\delta v) - F(u,\ v)\}, \text{ identically,}$$
$$= A + B, \text{ say.}$$

Choose a subsidiary function to express the effect of a change δu in $F(u, v)$ when v is unchanged, thus

$$\Phi(u, v) = F(u+\delta u, v) - F(u, v).$$

$$A = F(u+\delta u, v+\delta v) - F(u, v+\delta v) = \Phi(u, v+\delta v).$$

Now consider the change in Φ when u is kept constant, and v receives an increment δv. By the definitions of p. 101 and p. 192,

$$\Phi(u, v+\delta v) = \Phi(u, v) + \delta v\,\{\underset{u\text{ const.}}{D_v \Phi(u, v)} + \epsilon\}.$$

But $\Phi(u, v) = \delta u\,\{\underset{v\text{ const.}}{D_u F(u, v)} + \epsilon'\}$ and for all values of u contains the factor δu. Therefore an increment of $\Phi(u, v)$ contains the factor δu, i.e. $\underset{u\text{ const.}}{D_v \Phi(u, v)}$ contains the factor δu, and may be written $\delta u \times K$, where K is a finite varying quantity.

$$\therefore\ A = \Phi(u, v+\delta v) = \delta u\,\{\underset{v\text{ const.}}{D_u F(u, v)} + \epsilon'\} + \delta v\,\{\delta u \times K + \epsilon\}.$$

* This is often written $\frac{dy}{dx} = \frac{\partial y}{\partial u} \cdot \frac{du}{dx} + \frac{\partial y}{\partial v} \cdot \frac{dv}{dx}$, ∂ signifying partial differentiation and d complete differentiation.

Also $B = F(u+\delta u, v) - F(u, v)$ is the increment obtained in F by increasing v without altering u,

$$= \delta v\, \{\underset{u \text{ const.}}{D_v F(u, v)} + \epsilon''\}.$$

$$\therefore\ \delta y = A + B = \delta u\, \{\underset{v \text{ const.}}{D_u F^*} + \epsilon'\} + \delta v\, \{\underset{u \text{ const.}}{D_v F} + \epsilon''\}$$
$$+ \delta u\,.\,\delta v \times K + \epsilon\,.\,\delta r.$$

Divide by δx and proceed to the limit when δu, δv, δx, δy vanish.

$$\mathrm{L}^t \frac{\delta y}{\delta x} = \mathrm{L}^t \frac{\delta u}{\delta x}\{D_u F + \epsilon'\} + \mathrm{L}^t \frac{\delta v}{\delta x}\{D_v F + \epsilon''\}$$
$$+ \mathrm{L}^t \frac{\delta u}{\delta x} \times \mathrm{L}^t \delta v \times K + \epsilon\, \mathrm{L}^t \frac{\delta v}{\delta x}.$$

In limit ϵ, ϵ', ϵ'' may be neglected, and we have

$$\mathbf{D_x\, y = \underset{v\ \text{const.}}{D_u\, y} \,.\, D_x\, u + \underset{u\ \text{const.}}{D_v\, y} \,.\, D_x\, v.}$$

In particular, if $u = x$, $\mathbf{D_x\, y} = \underset{\mathbf{v\ const.}}{\mathbf{D_x\, y}} + \underset{\mathbf{x\ const.}}{\mathbf{D_v y}}\,.\,\mathbf{D_x\, v}$, where y is a function of v and x, and v is a function of x.

If we have the locus $f(x, y) = 0$, write z for $f(x, y)$. Then z is constantly zero and has no increment.

$$\therefore\ 0 = D_x z = \underset{y \text{ const.}}{D_x f} + \underset{x \text{ const.}}{D_y f}\,.\,D_x\, y,$$

and
$$\mathbf{D_x\, y} = -\frac{\mathbf{D_x\, f\ (y\ const.)}}{\mathbf{D_y\, f\,(x\ const.)}}.$$

The tangent at $(x_1\, y_1)$ to $f(x\, y) = 0$ may be written

$$(x - x_1)\, D_x f + (y - y_1)\, D_y f = 0,$$

when x_1 and y_1 are written for x, y in the derived functions.

Thus in

$$ax^2 + 2\,hxy + by^2 = 1,\ \underset{y \text{ const.}}{D_x f} = 2\,ax + 2\,hy,\ \underset{x \text{ const.}}{D_y f} = 2\,hx + 2\,by\,;$$

the tangent is $(x - x_1)(ax_1 + hy_1) + (y - y_1)(hx_1 + by_1) = 0$,

$$axx_1 + h\,(xy_1 + yx_1) + byy_1 = ax_1^{\,2} + 2\,hx_1 y_1 + by_1^{\,2} = 1,$$

since $x_1\, y_1$ is on the curve.

---

* $D_x f$ is used for $D_x f(x)$, and generally f, or a similar letter, is used as an abbreviation for the function.

By a similar line of reasoning we may show that if y is a function of any finite number of variables $\alpha, \beta, \gamma, \ldots$, all of which are functions of a variable t, then

$$D_t y = \frac{\partial y}{\partial \alpha} \cdot D_t \alpha + \frac{\partial y}{\partial \beta} D_t \beta + \frac{\partial y}{\partial \gamma} D_t \gamma + \ldots,$$

where $\frac{\partial y}{\partial \alpha}$ (for example) is obtained by differentiating y with respect to α, while $\beta, \gamma, \ldots$ are regarded as constant. This formula is of great importance in many applications of the calculus.

In particular, if $\alpha = t$, the first term on the right becomes $\frac{\partial y}{\partial t}$.

Maxima and Minima of a Function of two Independent Variables.

Let $y = F(u, v)$, and therefore

$$\delta y = \underset{v \text{ const.}}{D_u F} . \delta u + \underset{u \text{ const.}}{D_v F} \delta v + \text{quantities ultimately negligible.}$$

y is said to be a maximum (minimum) when any small change in u or v decreases (increases) its value.

Then unless $D_u F = 0$ there cannot be a maximum or minimum; for otherwise increasing u without changing v would, if $D_u F$ were positive, result in an increase in y, and, if $D_u F$ were negative, in a decrease. Hence we cannot have a maximum or minimum unless $\underset{v \text{ const.}}{D_u F} = 0$, and similarly unless $\underset{u \text{ const.}}{D_v F} = 0$. In cases where we know *a priori* that a maximum or minimum exists, these equations are often sufficient to determine its position, and it is often possible by substitution to decide whether it is maximum or minimum. In other cases further and more complicated equations are needed.

EXAMPLES.—(i) If $y = u^3 + v^3 + 3auv$.

$$\underset{v \text{ const.}}{D_u y} = 3u^2 + 3av = 0, \text{ and } \underset{u \text{ const.}}{D_v y} = 3v^2 + 3au = 0.$$

These are satisfied by $u = v = 0$, and $u = v = -a$.

Taking the latter solution, if $u = -a + h$, $v = -a + k$,

$$y = (-a + h)^3 + (-a + k)^3 + 3a(-a + h)(-a + k)$$
$$= a^3 - 3a(h^2 + k^2 - hk) + h^3 + k^3.$$

The coefficient of $-3a = \left(h - \frac{k}{2}\right)^2 + \frac{3k^2}{4}$ and is always positive.

A small variation from the values $u = v = -a$ then always results in a diminution of y, since $h^3 + k^3$ may be neglected in comparison with $h^2 + k^2 - hk$.

Hence a maximum value of y is when $u = v = -a$, and $y = a^3$.

(ii) The surface of a six-sided rectangular solid of given volume is least when the edges are equal.

Here there is evidently a minimum.

Let V be the volume, u, v, w the edges, and $2y$ the surface.

$$V = uvw,$$

$$y = uv + vw + wu = uv + V\left(\frac{1}{u} + \frac{1}{v}\right), \text{ where } V \text{ is constant,}$$

$$0 = \underset{v \text{ const.}}{D_u y} = v - \frac{V}{u^2}; \quad \therefore\ u^2 v = uvw, \text{ and } u = w.$$

Similarly $v = w$.

(iii) Find the minimum value of

$$au^2 + 2huv + bv^2 + 2gu + 2fv + c.$$

Necessary conditions are

$$2au + 2hv + 2g = 0 = 2hu + 2bv + 2f.$$

Substituting the values thus found, say $\bar{u}$, $\bar{v}$, the expression $= \dfrac{\Delta}{C}$ (pp. 176–7).

If $\bar{u} + u_1$ and $\bar{v} + v_1$ are written for u and v, the expression becomes

$$\frac{\Delta}{C} + au_1^2 + 2hu_1v_1 + bv_2^2,$$

and the increment is positive if $ab > h^2$ and a is positive. In this case we have found a minimum. If $ab > h^2$ and a negative we have a maximum.

If $ab \ngtr h^2$ we can proceed no further by this method.

(iv) Show that $z = Ae^{-(hx^2+ky^2)}$ has a maximum, A, at $x = y = 0$, if h and k are positive.

SECTION VIII

IMAGINARY AND COMPLEX QUANTITIES *

IN the solution of the quadratic equation $x^2+2x+7=0$ we obtain $(x+1)^2=-6$, and we cannot proceed further without using a quantity (d) such that $d\times d=-6$. So far no meaning has been attached to such a quantity. In algebra and trigonometry as applied to arithmetic and mensuration the letters used stand for quantities known in the physical universe, whose squares are positive; as soon as the notation of algebra is begun, the statement $(-a)\times(-a)=a^2$ is either proved when a is regarded as a physical measurement, or assumed as a law of operation or rule of interpretation or convention when algebra is treated as a purely abstract science.

A very great extension of mathematics, in the end of enormous practical importance, has been made by introducing a second set of symbols which are subject to the convention that the values obtained by a process analogous to squaring are negative. These symbols are said to represent *imaginary* quantities, that is, quantities that are the subject of imagination or thought only, as opposed to *real* quantities whose application to physical measurements is direct.

It is open to us to make any rules, not inconsistent with each other, for the algebra of imaginary quantities and to introduce any symbols (of the same kind as $\times$, $\div$, $>$) we please. We shall, however, be guided by the general rule that symbols and operations are to have as nearly as possible the same meanings whatever may be the quantities they affect, as we were in assigning meanings to fractional indices (p. 2) and to angles of any magnitude and to their ratios (p. 40). In such cases we can either deduce rules in accordance with a convention, which we

* The treatment of this subject as far as p. 233 differs completely in definitions, order, and method from that generally given in text-books.

decide that the symbols shall satisfy, as with indices, or make rules and then show that they are consistent with the convention. In the present case it is convenient to combine these processes.

Definition of an Imaginary Quantity.

Let $\mathfrak{O}(a)$ signify that an operation has been performed on a, producing a quantity a_ι, such that when a similar operation is performed on a_ι the result is $-a$. Thus $\mathfrak{O}(a_\iota) = -a$, and $\mathfrak{O}\{\mathfrak{O}(a)\} = -a$. Then a_ι is called an imaginary quantity.

The operation, $\mathfrak{O}$, is an hypothetical or imaginary operation, and cannot be carried out by any of the laws of algebra hitherto used.

$$\mathfrak{O}(1) = 1_\iota, \text{ and } \mathfrak{O}\{\mathfrak{O}(1)\} = -1.$$

Rules defining the Use of $+$, $-$, $\times$, $\div$ with Imaginary Quantities.

Addition and subtraction of imaginary quantities.

Rules. $\mathbf{a}_\iota \pm \mathbf{b}_\iota = \mathbf{c}_\iota$, **where** $\mathbf{c} = \mathbf{a} \pm \mathbf{b}$, **and a, b, c are real,** (i)

$$\mathbf{a}_\iota + \mathbf{b}_\iota = \mathbf{b}_\iota + \mathbf{a}_\iota, \text{ and } -\mathbf{a}_\iota = (-\mathbf{a})_\iota.$$

Definition. Imaginary zero is defined by $0_\iota = a_\iota - a_\iota$.

Multiplication and division of imaginary quantities by real quantities.

If b is a real positive integer,

$$\mathbf{a}_\iota \times \mathbf{b} = a_\iota + a_\iota + \dots \ (b \text{ terms}) = (a + a + \dots)_\iota \text{ by (i).}$$
$$= \mathbf{d}_\iota, \text{ where } \mathbf{d} = \mathbf{ab}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Rule. $\mathbf{f}_\iota = \mathbf{a}_\iota \div \mathbf{b}$, where b is a real positive integer, is taken so as to satisfy the condition that $f_\iota \times b = a_\iota$; then, by (ii),

$$a = bf, \text{ and } \mathbf{f} = \frac{\mathbf{a}}{\mathbf{b}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

Hence $a_\iota \times \dfrac{p}{q}$, where p and q are real integers,

$$= (a_\iota \times p) \div q = g_\iota, \text{ where } g = \frac{ap}{q}.$$

We have then generally, $a_\iota \times m = (ma)_\iota$ and $a_\iota \div m = \left(\dfrac{a}{m}\right)_\iota$, where m is any positive real commensurable quantity.

For operation with a negative multiplier, we decide that

$$a_\iota \times (m) + a_\iota \times (-m) = 0_\iota$$

But $0_\iota = (ma)_\iota - (ma)_\iota = a_\iota \times m + (-ma)_\iota$, from (i).

Then $a_\iota \times (-m) = (-ma)_\iota$.

It readily follows that $a_\iota \div (-m) = \left(-\frac{a}{m}\right)_\iota$.

If m is incommensurable, multiplication and division by m are defined by deciding that the preceding equations shall still be true.

Write ι *for* 1_ι.

Then, in particular, $\mathbf{m}_\iota = (m \times 1)_\iota = 1_\iota \times m = \iota \times \mathbf{m}$. . . (iv)

Multiplication and Division by Imaginary Quantities.

Particular Rule. $\mathbf{k} \times \iota = \mathbf{k}_\iota$; that is,

$$\mathbf{k} \times \iota = \mathbf{k} \times \mathbf{1}_\iota = \mathbf{k}_\iota = \mathfrak{D}(\mathbf{k}). \quad . \quad . \quad . \quad . \qquad \text{(v)}$$

In words, $\times\, \iota$ signifies the performance of the operation $\mathfrak{D}$.

If k is real, $k \times \iota = k_\iota = \iota \times k$ by (iv).

General Rule. $\mathbf{k} \times \mathbf{a}_\iota = (\mathbf{k} \times \mathbf{1}_\iota) \times \mathbf{a}$, a being real, . . (vi)

$$= \mathfrak{D}(\mathbf{k}) \times \mathbf{a}.$$

If k is real, it follows that

$$\mathbf{k} \times \mathbf{a}_\iota = \mathbf{k}_\iota \times \mathbf{a} = (\mathbf{ak})_\iota \text{ from (iv)}. \quad . \quad . \quad . \qquad \text{(vii)}$$

If $k = a_\iota$ (a real), $a_\iota \times a_\iota = \mathfrak{D}(a_\iota) \times a = -a \times a = -a^2$.

In particular, $\iota \times \iota = 1_\iota \times 1_\iota = -1^2 = -1$.

[By a loose analogy with the process of extraction of a square root for real quantities, $\sqrt{-1}$ is often written for ι.]

Similarly, $\mathbf{b}_\iota \times \mathbf{a}_\iota = \mathfrak{D}(\mathbf{b}_\iota) \times \mathbf{a} = -\mathbf{b} \times \mathbf{a} = -\mathbf{ab}$. . . (viii)

As with real quantities, division by a_ι is taken so as to annul the effect of multiplying by a_ι.

Rule. $(k \div a_\iota) \times a_\iota = k$, where k is real or imaginary.

$$\therefore \ \mathfrak{D}(k \div a_\iota) \times a = k \text{ by (vi)}.$$

$$\therefore \ \mathfrak{D}\{\mathfrak{D}(k \div a_\iota)\} = \mathfrak{D}\left(\frac{k}{a}\right).$$

If $k = b$ (b real), this becomes $-(b \div a_\iota) = \left(\frac{b}{a}\right)_\iota$,

$$\text{i.e. } \mathbf{b} \div \mathbf{a}_\iota = -\left(\frac{\mathbf{b}}{\mathbf{a}}\right)_\iota; \quad . \quad . \quad . \quad . \quad . \qquad \text{(ix)}$$

if $k = b_\iota$, $-(b_\iota \div a_\iota) = \mathfrak{D}\left(\frac{b_\iota}{a}\right) = \mathfrak{D}\left(\frac{b}{a}\right)_\iota$ by (iii),

and

$$\mathbf{b}_\iota \div a_\iota = \frac{\mathbf{b}}{\mathbf{a}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(x)}$$

It follows from the above definitions and rules that a_ι can be taken in all the processes of addition, multiplication, subtraction, and division as if it was $a \times \iota$, where ι is used just as if it was a real quantity. Thus

$$a\iota \pm b\iota = (a \pm b)\,\iota, \qquad b \times a\iota = a\iota \times b = ab\iota,$$

$$a\iota \div b = \frac{a\iota}{b}, \qquad a\iota \times b\iota = a \times b \times \iota \times \iota = -ab,$$

$$a\iota \div b\iota = \frac{a}{b}, \quad \text{and} \quad b \div a\iota = \frac{b\iota}{a\iota\iota} = -\frac{b\iota}{a}$$

give the results detailed above.

If n is a positive integer, $(a_\iota)^n = a_\iota \times a_\iota \times$ to n factors
$= a^n \iota^n$,
where $\iota^2 = -1$, $\iota^3 = -\iota$, $\iota^4 = +1$, $\iota^5 = \iota$, $\iota^6 = -1$, &c.

We have given no meaning as yet to $(a_\iota)^n$, where n is not a positive integer; this is examined on p. 230, below.

We have given no meaning to $a_\iota \pm b$, where b is real; this is dealt with in the paragraphs immediately following.

The operation $\boldsymbol{\iota}$ has a close analogy in geometry.

Let a distance a be measured to right and left from O, as in co-ordinate geometry, and let $OA = a$, $OA' = -a$, as in Fig. 87.

OA' can be obtained from OA either by rotating OA through 180°, or by rotating through 90° to OB and then again through 90° to OA'.

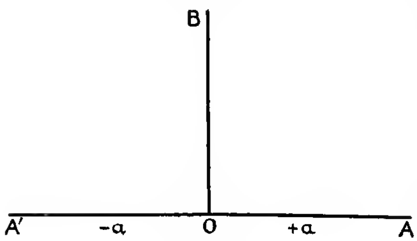


Fig. 87.

The complete operation is equivalent to multiplying by -1. The operation of rotating through a right angle is such that if repeated the result is multiplication by -1. Rotating through a right angle is therefore similar to performing the operation $\boldsymbol{\iota}$.*

* This conception is used by modern mathematicians to obtain two algebraic dimensions without using spatial relations.

Following out this clue, take two axes OX, OY at right angles, with scales from $-\infty$ to $+\infty$.

Represent any real quantity a by a length OA on OX, and any imaginary quantity b_ι by a length OB ($= b$ on the scale) on OY.

A new quantity can now be defined in relation to any two quantities, one real and one imaginary, as follows:

Mark off $OM = x$ along OX, and $ON = y$ along OY, where

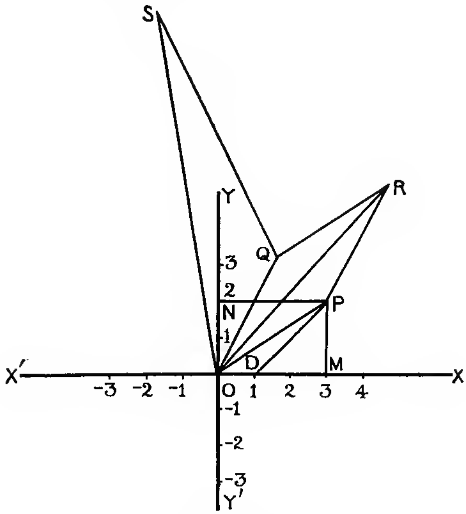


FIG. 88.

x and y_ι are any real and imaginary quantities. Complete the rectangle $OMPN$. Join OP. Let $\angle XOP = \theta$, and $OP = r$.

Then $\theta = \tan^{-1}\frac{y}{x}$ and $r = \sqrt{x^2+y^2}$. Regard r as always positive, and θ as any (real) angle, positive or negative.

DEFINITION. A complex quantity is one which can be completely represented by the line (or vector*) OP, thus constructed with reference to the real quantity x and the imaginary quantity y_ι; this quantity is written (x, y) or (r, θ).

* Vector is used to denote a line drawn in an assigned direction and of assigned length.

r is called the *modulus* (or measure) and θ the *amplitude*,* x the real part, y_ι the imaginary part.

Rules defining the use of +, −, ×, ÷ with Complex Quantities.

These rules are so chosen that when the y's are zero they are identical with the rules of real quantities, and when the x's are zero with those of imaginary quantities.

Let OP (r_1, θ_1) and OQ (r_2, θ_2) be the same quantities as (x_1, y_1), (x_2, y_2) respectively.

Rule. $$(\mathbf{x}_1, \mathbf{y}_1) + (\mathbf{x}_2, \mathbf{y}_2) = (\mathbf{x}', \mathbf{y}'),$$

where $$\mathbf{x}' = \mathbf{x}_1 + \mathbf{x}_2 \text{ and } \mathbf{y}' = \mathbf{y}_1 + \mathbf{y}_2. \quad . \quad . \quad . \quad . \quad \text{(xi)}$$

Complete the parallelogram $OPRQ$. Then OR is (x', y') or (r', θ').

Here $$r' = \sqrt{\{r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_2 - \theta_1)\}},$$

and $$\theta' = \tan^{-1}\left(\frac{r_1 \sin\theta_1 + r_2 \sin\theta_2}{r_1 \cos\theta_1 + r_2 \cos\theta_2}\right).$$

Hence r' is always less than $r_1 + r_2$, unless $\theta_1 = \theta_2$.

If any of the quantities x_1, x_2, y_1, y_2 are negative, subtraction is involved, but no new rule is needed.

If $y_1 = 0 = y_2$ we get the ordinary rule for real quantities. If $x_1 = 0 = x_2$ we get rule (i) for imaginary quantities.

Rule. $$\begin{matrix} OP & & OQ & & OS \\ (\mathbf{r}_1, \boldsymbol{\theta}_1) & \times & (\mathbf{r}_2, \boldsymbol{\theta}_2) & = & (\mathbf{r}', \boldsymbol{\theta}'), \end{matrix}$$

where $$\mathbf{r}' = \mathbf{r}_1 \times \mathbf{r}_2 \text{ and } \boldsymbol{\theta}' = \boldsymbol{\theta}_1 + \boldsymbol{\theta}_2. \quad . \quad . \quad . \quad . \quad \text{(xii)}$$

S may be obtained in Figure 88 as follows:

Let $OD = 1$ on OX. Join DP.

Construct a triangle OQS, similar to ODP. Then

$$\angle SOX = \angle SOQ + \angle QOD = \theta_1 + \theta_2 = \theta',$$

and $OS : OQ :: OP : OD$, i.e. $OS \times 1 = r_1 r_2$, and $\therefore$ $OS = r'$.

* Also called the *argument*. In astronomy amplitude is used as the angular distance of the position of a rising (or setting) star from the point on the horizon due E. (or W.) of the observer. Thus XX' may be regarded as the E. and W. line.

If OS be (x', y'),

$$x' = r' \cos\theta' = r_1 r_2 (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) = x_1 x_2 - y_1 y_2,$$

and $$y' = r' \sin\theta' = x_1 y_2 + x_2 y_1.$$

If $y_1 = y_2 = 0$, then $y' = 0$ and $r' = x' = x_1 x_2 = r_1 r_2$, as with real numbers.

If $x_1 = x_2 = 0$, then $y' = 0$ and $x' = -y_1 y_2$, which corresponds with (viii) for multiplication of imaginary numbers.

If $y_1 = 0 = x_2$, then $x' = 0$ and $y' = x_1 y_2 = y_2 x_1$, as in (ii) and (vii).

If $r_2 = 1$, $\theta_1 = 0$, and $\theta_2 = \frac{1}{2}\pi$, then $\theta' = \frac{1}{2}\pi$, $x' = 0$, $y' = r' = r_1 = x_1$, and we have the process called **I** above.

If $\theta_2 = 0$, then $\theta' = \theta$, and we have simply a prolongation of OP.

Division is to annul the effect of multiplication.

$$\{(r_1, \theta_1) \div (r_2, \theta_2)\} \times (r_2, \theta_2) = (r_1, \theta_1).$$

But by (xii), $\left(\frac{r_1}{r_2}, \theta_1 - \theta_2\right) \times (r_2, \theta_2) = (r_1, \theta_1)$.

Rule. **Hence the quotient of (r_1, θ_1) divided by (r_2, θ_2) is a complex quantity, whose modulus is $\frac{r_1}{r_2}$ and amplitude $(\theta_1 - \theta_2)$** . . . . (xiii)

It follows that

$$\left(\frac{1}{r}, -\theta\right) = 1 \div (r, \theta), \text{ which may be written } \frac{1}{(r, \theta)}. \quad \text{(xiv)}$$

Indices.

Rule. When n is a real positive integer,

$$(r, \theta)^n = (r, \theta) \times (r, \theta) \times \ldots \ (n \text{ factors});$$

but $$(r, \theta)^2 = (r^2, 2\theta) \text{ by (xii)},$$

and it is easily seen that $(r, \theta)^n = (r^n, n\theta)$ . . . . . . (xv)

It follows that if m is also a real positive integer,

$$(r, \theta)^n \times (r, \theta)^m = (r^n, n\theta) \times (r^m, m\theta) = (r^{n+m}, \overline{m+n}\,\theta). \quad \text{(xvi)}$$

We shall obtain the meaning of fractional and negative indices by making the rule that the resulting quantities are to satisfy (xvi). (Compare pp. 2, 3.)

Let p and q be real positive integers.

Then $\left(r^{\frac{p}{q}}, \frac{p}{q}\theta\right)^q = \left\{\left(r^{\frac{p}{q}}\right)^q, p\theta\right\}$ by (xv)

$= (r^p, p\theta), = OP$, say, $= (r, \theta)^p$ by (xv).

Here $\left(r^{\frac{p}{q}}, \frac{p}{q}\theta\right)$ is called a q^{th} root of the quantity OP.

We can obtain q different roots to satisfy this equation as follows:

The quantity $(r^p, p\theta + 2k\pi)$ is represented by one and the same vector (say OP), where k is zero or any integer, and this formula includes all possible values of θ for this vector.

But by the equation just given,

$$\left(r^{\frac{p}{q}}, \frac{p\theta + 2k\pi}{q}\right)^q = (r^p, p\theta + 2k\pi) = (r^p, p\theta).$$

The equation is therefore satisfied by a complex quantity, with modulus $r^{\frac{p}{q}}$ (always real and positive and therefore not ambiguous) and with amplitude any of the angles $\frac{p}{q}\theta$, $\frac{p\theta + 2\pi}{q}$, $\frac{p\theta + 4\pi}{q}$ .... The $(q+1)^{\text{th}}$ term is $\left(\frac{p}{q}\theta + 2\pi\right)$, which has the same vector as $\frac{p}{q}\theta$, and the others are repeated in the same way. It is easy to see that negative values of k give no new vectors.

We have now obtained q distinct roots of OP, and may write

$$(\mathbf{r}, \theta)^{\frac{\mathbf{p}}{\mathbf{q}}} = \left(\mathbf{r}^{\frac{\mathbf{p}}{\mathbf{q}}}, \frac{\mathbf{p}}{\mathbf{q}}\theta + \frac{2\mathbf{k}\pi}{\mathbf{q}}\right), \text{ or } (\mathbf{r}, \theta)^{\mathbf{n}} = \left(\mathbf{r}^{\mathbf{n}}, \mathbf{n}\theta + \frac{2\mathbf{k}\pi}{\mathbf{q}}\right),$$

where n is any commensurable *positive* quantity, q the denominator of n if n is fractional, and k is zero or any integer. (xvii)

If n is *zero*, the convention of equation (xvi) gives,

$$(r, \theta)^1 \times (r, \theta)^0 = \{r^{1+0}, (1+0)\theta\} = (r, \theta)^1,$$

and $(r, \theta)^0$ is to be taken as the real quantity 1; here $x = 1 = r$, $y = 0 = \theta$.

If n is a *negative* commensurable quantity, let $n = -n'$.

Then, by the convention,

$$(r, \theta)^n \times (r, \theta)^{n'} = (r, \theta)^{n+n'} = (r, \theta)^0 = 1.$$

$$\therefore\ (r, \theta)^n = \frac{1}{(r, \theta)^{n'}} = \frac{1}{\left(r^{n'}, \left(n'\theta + \frac{2k\pi}{q}\right)\right)} \text{ by (xvii)}$$

$$= \left(r^{-n'}, -\left(n'\theta + \frac{2k\pi}{q}\right)\right) \text{ by (xiv)}$$

$$= \left(r^n, n\theta + \frac{2k\pi}{q}\right),$$

since $+k$ and $-k$ have the same meaning, viz. any integer, positive or negative, or zero.

Hence the statement (xvii) may be extended to include all commensurable values of n, positive or negative, including $n = 0$.

This result is known as De Moivre's Theorem.

No meaning has yet been given to an incommensurable, imaginary, or complex index for complex quantities.

If $x = 0$ we have rules for powers of imaginary quantities, contained in the statement $y_\iota^n = \left(y^n, \frac{1}{2}n\pi + \frac{2k\pi}{q}\right)$, since here $r = y$ and $\theta = \frac{1}{2}\pi$.

Example. Compare this with ι^n when n is a positive integer.

If the whole of the previous analysis is written with $r = 1$, all the points are restricted to a circle of unit radius and there can be no variation of the modulus.

$$\therefore\ (1, \theta)^n = \left(1, n\theta + \frac{2k\pi}{q}\right).$$

If $\theta = 0$, and $n = \frac{1}{q}$, where q is a positive integer, we have that the q^{th} roots of a real quantity x are $\left(x^{\frac{1}{q}}, \frac{2k\pi}{q}\right)$. This is best illustrated by the following paragraph.

The n nth Roots of Unity.

$1^{\frac{1}{n}} = (1, 0)^{\frac{1}{n}} = \left(1, 0 + \frac{2k\pi}{n}\right)$. Hence the complex quantities, whose amplitudes are $0, \frac{2\pi}{n}, \frac{4\pi}{n}, \ldots 2\pi - \frac{2\pi}{n}$ and whose moduli are 1, are n different n^{th} roots of unity.

If n is odd, there is only one real root (1, 0), and the amplitudes of the other roots are the $\frac{n-1}{2}$ pairs $\pi \pm \frac{\pi}{n}$,

$$\pi \pm \frac{3\pi}{n}, \ldots \pi \pm \frac{n-2}{n}\pi.$$

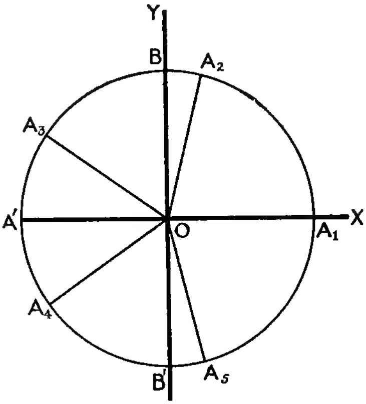


FIG. 89.

E.g. if $n = 5$, the roots are OA_1, OA_2, OA_3, OA_4, OA_5, where the A's are the angular points of a regular pentagon, as in Figure 89.

If n is even, the roots are 0, π, that is, the real quantities $+1$ and -1, and the remaining amplitudes are

$$\pi \pm \frac{2\pi}{n}, \ \pi \pm \frac{4\pi}{n}, \ \ldots \ \pi \pm \left(\frac{n-1}{n}\right)\pi.$$

The fourth roots of unity are OA_1, OB, OA', OB' in the figure, that is, the real quantities ± 1, and the imaginary quantities $\pm \iota$.

EXAMPLES. Find the 7th roots of 128.

Find the 6th roots of 2037, using logarithms.

We found that the operation ⨅ or $\times \iota$ is analogous both to an imaginary process of taking the square root of -1 (p. 225) and to rotation through a right angle. We can now see that the result of taking any (the n^{th}) integral root of -1 is a complex quantity $\left(-1, \frac{\pi}{n}\right)$, obtained by rotating through $\frac{1}{n}^{\text{th}}$ of two right angles.

The introduction of the ideas of imaginary and complex quantities has enabled us to attach a meaning to roots of any order of any real quantity.

We have now attached meanings to $+$, $-$, $\times$, $\div$ and a real index when applied to complex and imaginary quantities so as to satisfy the conditions that they are definite and consistent with each other, and so that the laws for complex quantities become those of real quantities when the y's are zero and those of imaginary quantities when the x's are zero. We proceed to show that the laws satisfy a further remarkable and important condition.

Put alongside each other the rules of addition and of multiplication of real and of complex quantities.

Real: $(a_1+b_1)+(a_2+b_2) = (a_1+a_2)+(b_1+b_2)$.

Complex: $(x_1, y_1)+(x_2, y_2) = (\overline{x_1+x_2}, \overline{y_1+y_2})$.

Real: $(a_1+b_1)\times(a_2+b_2) = (a_1 a_2+b_1 b_2)+(a_1 b_2+a_2 b_1)$.

Complex: $(x_1, y_1)\times(x_2, y_2) = (\overline{x_1 x_2 - y_1 y_2}, \overline{x_1 y_2 + x_2 y_1})$.

If now we write the complex quantities (x_1, y_1), &c., as $x_1+\iota y_1$, &c., where + simply means that x_1 and ιy_1 are united in a complex quantity, then we find that the rules of addition and multiplication can be obtained as if this + signified addition and the rules of real quantities were applied to ιy.

For then we should have

$$(x_1+\iota y_1)+(x_2+\iota y_2) = x_1+x_2+\iota y_1+\iota y_2 = (\overline{x_1+x_2}, \overline{y_1+y_2}),$$

and
$$(x_1+\iota y_1)\times(x_2+\iota y_2) = x_1 x_2+\iota^2 y_1 y_2+x_1 y_2+x_2 y_1$$
$$= (\overline{x_1 x_2 - y_1 y_2}, \overline{x_1 y_2+x_2 y_1}), \text{ since } \iota^2 = -1.$$

The notation $x+\iota y$ is generally used; and $z = x+\iota y$ is taken to mean that z is the complex quantity whose real and imaginary parts are x and y_ι.

Both for real and for imaginary quantities subtraction and division are the processes which annul the effect of addition and multiplication respectively; and the meaning of indices has been deduced from the same convention for real quantities (p. 2) and complex (p. 229, equation xvi).

EXAMPLE. Find from (xiii) the result of $(x_1, y_1) \div (x_2, y_2)$, and verify that it satisfies the above method.

It follows that with this conventional use of + in complex quantities, we may apply all the rules and processes of algebra up to the solution of equations, ratio, progressions, and indices, whatever the letters stand for, but not as yet to logarithms, limits, or series, since these words have not been defined in connexion with imaginary and complex quantities.

We may now remove the restriction that the letters used in algebra mean real quantities, being prepared to interpret any letter as meaning either a real, imaginary, or complex quantity according to the conditions of the problem. By this means we obtain an enormous extension of the power of mathematical operations; we can perform all processes, without reference to there being a physical or concrete interpretation for them. If the data of a problem are physical and our processes correct, we are bound to obtain a result in real quantities, if such a result exists. If there is no real, but only a complex or imaginary solution, we should learn that the hypothesis was physically impossible. For example, the times at which a body projected upwards with velocity v will be at a height h are given by the equation $h = vt - \frac{1}{2}gt^2$. This gives $\left(t - \frac{v}{g}\right)^2 = \frac{v^2 - 2hg}{g^2}$. If $v^2 \nless 2hg$, we obtain a real value for t; otherwise t is the complex quantity $\frac{v}{g} + \iota y$, where $y^2 = \frac{2hg - v^2}{g^2}$, and the body does not in fact reach the height h. Thus we can give a solution of such a quadratic as stated at the beginning of this section. The sum and product of the roots are $\frac{2v}{g}$ and $\frac{2h}{g}$ and are real, whether the solution is real or not.

The sequel will show that many important results can be readily obtained by the use of complex quantities, for which no easy proof is available without them; and the student may take on trust the general statement that the great part of advanced mathematical analysis, which has innumerable practical results in mechanics and physics, would be impossible, or so cumbersome as to be impracticable, if all letters were restricted to mean real quantities.

The following examples are given to show further the relation between the various ways of expressing a complex quantity, viz. $z = x + \iota y = (x, y) = (r, \theta)$, and the use of the z notation.

We have $z^n = (r^n, n\theta)$; $b \times z = (b, 0) \times (r, \theta) = (rb, \theta)$;

$$a + bz + cz^2 = (a, 0) + (br, \theta) + (cr^2, 2\theta),$$

where a, b, c are real, and these complex quantities can be added and expressed as one; or

$$a + b(x + \iota y) + c(x + \iota y)^2 = a + bx + c(x^2 - y^2) + \iota(by + 2cyx).$$

Similarly, any rational integral function, $f(z)$, of z can be expressed as one complex quantity.

EXAMPLE. Work out $a+bz+cz^2$, and mark z and the result on the diagram, when $z = (\frac{1}{2}, \frac{1}{3}\pi)$, $a = 3$, $b = -2$, and $c = 4$.

If $\qquad f_1(z) = (r_1, \theta_1)$ and $f_2(z) = (r_2, \theta_2)$,

then $\qquad f_1(z) \div f_2(z) = \left(\frac{r_1}{r_2}, \theta_1 - \theta_2\right)$ from equation (xii),

and thus a fraction whose numerator and denominator are rational integral functions of z can be expressed as one complex quantity.

If $f(z)$ is a single-valued function of z (that is, is uniquely determinate when z is given) and is expressible by two different processes as $a+b\iota$ and $c+d\iota$ for the same value of z, then $a=c$ and $b=d$. For if OP represents the function on the diagram of complex quantities, a and c are both its projections on OX, and b and d on OY.

E.g. $(1+\cos\theta+\iota\sin\theta)^2$

$$= 1+2\cos\theta+\cos^2\theta-\sin^2\theta+2\iota\sin\theta\,(1+\cos\theta),$$

by direct multiplication, and

$$= (2\cos^2\tfrac{1}{2}\theta+2\iota\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta)^2$$

$$= 4\cos^2\tfrac{1}{2}\theta\,(\cos\tfrac{1}{2}\theta+\iota\sin\tfrac{1}{2}\theta)^2 = 4\cos^2\tfrac{1}{2}\theta\,(\cos\theta+\iota\sin\theta)$$

by De Moivre's theorem. Hence

$$1+2\cos\theta+\cos^2\theta-\sin^2\theta = 4\cos^2\tfrac{1}{2}\theta\,.\cos\theta,$$

and $\qquad 2\sin\theta\,(1+\cos\theta) = 4\cos^2\tfrac{1}{2}\theta\sin\theta,$

as can of course be proved directly.

If $f(z) = a+b\iota = 0$ for any value of z, then $a=0$ and $b=0$, for if either a or b have any value, $a+b\iota$ is represented by some line OP, not by O.

E.g. if $z = (2, \frac{1}{3}\pi)$

$$\frac{1+z^2}{1+2z+3z^2} = \frac{1+(4, \frac{2}{3}\pi)}{1+2(2, \frac{1}{3}\pi)+3(4, \frac{2}{3}\pi)}, \text{ in the } (r, \theta) \text{ notation}$$

$$= \frac{(1+4\cos\frac{2}{3}\pi,\ 4\sin\frac{2}{3}\pi)}{(1+4\cos\frac{1}{3}\pi+12\cos\frac{2}{3}\pi,\ 4\sin\frac{1}{3}\pi+12\sin\frac{2}{3}\pi)},$$

in the (x, y) notation,

$$= \frac{(-1,\ 2\sqrt{3})}{(-3,\ 8\sqrt{3})}, \text{ in the } (x, y) \text{ notation,}$$

$$= \frac{(\sqrt{13}, \tan^{-1} - 2\sqrt{3})}{(\sqrt{201}, \tan^{-1} - 8/\sqrt{3})}, \text{ in the } (r, \theta) \text{ notation},$$

$$= \left(\sqrt{\frac{13}{201}}, \tan^{-1}\frac{2\sqrt{3}}{51}\right), \text{ in the } (r, \theta) \text{ notation},$$

$$\text{or } \frac{1+z^2}{1+2z+3z^2} = \frac{1+(1+\iota\sqrt{3})^2}{1+2(1+\iota\sqrt{3})+3(1+\iota\sqrt{3})^2} = \frac{-1+\iota 2\sqrt{3}}{-3+\iota 8\sqrt{3}}$$

$$= \frac{(1-\iota 2\sqrt{3})(3+\iota 8\sqrt{3})}{(3-\iota 8\sqrt{3})(3+\iota 8\sqrt{3})} = \frac{51+\iota 2\sqrt{3}}{201}$$

$$= \sqrt{\frac{13}{201}}(\cos\theta + \iota\sin\theta), \text{ where } \tan\theta = \frac{2\sqrt{3}}{51}.$$

Fortunately, it is not often necessary to go through either of these processes.

Conjugate Complex Quantities.

(x, y) or (r, θ) and $(x, -y)$ or $(r, -\theta)$ are said to be conjugate.

Their sum, $2x$, and product, x^2+y^2, are both real. [This should be worked in both notations and shown on a diagram.]

If $f(z)$ is any function for which we can perform the operations necessary to express it as one complex quantity, and all the letters and numbers it involves other than z are real, and

$$\textbf{if} \quad \mathbf{f(z) = f(x+\iota y) = x'+\iota y'},$$
$$\textbf{then} \quad \mathbf{f(x-\iota y) = x'-\iota y'}.$$

For, let P be (x, y) and Q be (x', y'). Draw PP', QQ' parallel to OY so that PP' and QQ' are bisected by OX.

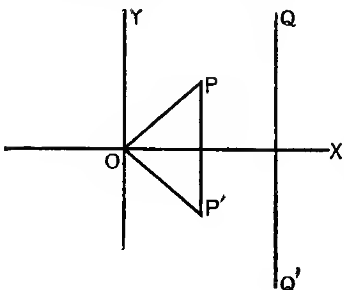


FIG. 90.

Then P' is $(x, -y)$.

The process of finding Q' from P' is exactly the same as that of finding Q from P with the diagram turned upside down. Hence Q' is $(x', -y')$.

Hence, P and P' being conjugate, Q and Q' are also conjugate.

EXAMPLE. $a+bz+cz^2$

$$= a+bx+c(x^2-y^2)+\iota(by+2cx), \text{ if } z = x+\iota y$$
$$= a+bx+c(x^2-y^2)-\iota(by+2cx), \text{ if } z = x-\iota y.$$

Digression on the Roots of an Equation.

If $f(z)$, a rational integral function of z, is (0, 0) when $z = \alpha + \iota\beta$, then $\alpha + \iota\beta$ is said to be a root of the equation $f(z) = 0$.

As on p. 86, divide $f(z)$ by

$$\{z-(\alpha+\iota\beta)\}\{z-(\alpha-\iota\beta)\} = z^2-2\alpha z+\alpha^2+\beta^2.$$

$$f(z) = Q\,.\,(z^2-2\alpha z+\alpha^2+\beta^2)+Rz+R',$$

where Q is a function two degrees lower than $f(z)$, and R and R' depend only on the coefficients of z in $f(z)$, and are real.

Write $\alpha + \iota\beta$ for z in this identity.

$$0 = f(\alpha+\iota\beta) = Q \times 0 + R(\alpha+\iota\beta)+R'.$$

$\therefore$ $R\alpha + R' + \iota R\beta = 0$, which is only possible (p. 235) if $R = 0$ and $R\alpha + R' = 0$.

$\therefore$ $R = 0 = R'$.

Now write $\alpha - \iota\beta$ for z in the identity.

$\therefore$ $f(\alpha-\iota\beta) = Q\,.\,\{0\}+0 = 0$, and $\alpha-\iota\beta$ is also a root.

Hence if (α, β) is a root of $f(z) = 0$, its conjugate $(\alpha, -\beta)$ is also a root.

[This may also be seen from the preceding paragraph, for if Q is at O, so is Q'.]

As on p. 87, it can now be shown that there cannot be more than n roots, real or imaginary, to an equation of the n^{th} degree.

If n is even, the roots may all be complex. If n is odd, $= 2m+1$, there cannot be more than m pairs of conjugate complex roots, and if m pairs can be found, there remains a real factor and a real root.

Coefficients of two finite rational functions can be equated as on p. 88, and as on p. 89 the sum of the roots, the sum of the products two together, &c., are related to the coefficients.

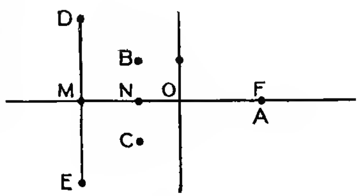


Fig. 91.

E.g. x^3-2x-4

$$= (x-2)\,(\overline{x+1}^2+1).$$

The roots of $x^3-2x-4 = 0$ are 2 (A), $-1+\iota$ (B), and $-1-\iota$ (C).

Their sum $= 0$ (coefficient of x^2). The sum of OB, OC, considered as vectors, is $2\,ON$, which $= -OA$.

Product two together

$$= 2 \times \underset{D}{(-1+\iota)} + 2 \times \underset{E}{(-1-\iota)} + \underset{F}{(-1+\iota)(-1-\iota)} = -2 \text{ (coeff. of } x).$$

(The sum of OD, OE, OF is $2\,OM$ and $OF = OM = -2$.)

Product of the three $= 4 =$ the absolute term $\times -1$.

THEOREM. Every equation, $f(z) = 0$, where f is a rational integral function of the n^{th} degree, has n roots, different or coincident.

[The following proof is only outlined. It is based on Burnside and Panton's *Theory of Equations*, 5th Ed., p. 260.]

Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$.

Let h be a small complex quantity.

Then $f(z+h) - f(z)$

$$= a_n \{(z+h)^n - z^n\} + a_{n-1}\{(z+h)^{n-1} - z^{n-1}\} + \dots + a_1 h$$

$$= h\{a_n n z^{n-1} + a_{n-1}(n-1) z^{n-2} + \dots + a_1\} + h^2 \times \text{ finite complex quantity.}$$

Let $f(z)$ be (r, θ). Let the coefficient of h be (r', θ').

Take h to be $(\rho, \pi + \theta - \theta')$.

Then $f(z+h) = f(z) + (\rho, \pi + \theta - \theta') . (r', \theta') + \rho^2 \times$ finite complex quantity.

$= (r, \theta) + (r'\rho, \pi + \theta)$* + a quantity which may be made small.

Let P represent z, P' $z+h$, Q $f(z)$, and Q' $f(z+h)$.

FIG. 92.

Now the sum of (r, θ) and $(r'\rho, \pi + \theta)$ is $(r - r'\rho, \theta)$. (Equation xi.)

Then if Q is not at O, P' a neighbouring position to P can be found, so that the resulting Q' is obtained by moving from Q towards O through a distance $r'\rho$, if quantities small in comparison to QQ'' are neglected; and ρ can be taken so that the inclusion of the smaller quantities still leaves $OQ' < OQ$.

Hence P can be theoretically moved by small finite steps to bring Q nearer O, till Q is within any small assignable distance of O.

* r' cannot be zero except for $\overline{n-1}$ values of z (p. 237), and since Q can be taken anywhere these positions can always be avoided.

Hence there is always a position of P which satisfies the equation $f(z) = 0$.

If the root is real, α, then $(z-\alpha)$ is a factor and the quotient is of the $\overline{n-1}^{\text{th}}$ degree.

If the root is complex, $\alpha + \iota\beta$, then $\alpha - \iota\beta$ is also a root, and the equation can be reduced 2 degrees.

A similar process can, theoretically, be applied till we come to the 2nd or 1st degree.

Thus the equation has n roots, real or complex.

It may happen that the root is repeated any number of times (up to $\overline{n-1}$) in the process.

Corollaries. Every equation of odd degree has one real root.

Every rational integral function of the $2m^{\text{th}}$ degree has m real quadratic factors. [These factors can be found when $m = 2$, but not in general for higher degrees.]

Two Important Series.

The binomial series for a positive integral index (p. 22) is simply the result of multiplication, and is unaltered for complex quantities.

Hence if n is integral,

$$\begin{aligned}\cos n\alpha + \iota \sin n\alpha &= (\cos\alpha + \iota\sin\alpha)^n \\ &= \cos^n\alpha - {}_nC_2 \cos^{n-2}\alpha \sin^2\alpha + {}_nC_4 \cos^{n-4}\alpha \sin^4\alpha - + \ldots \\ &\qquad + \iota\,(n \cos^{n-1}\alpha \sin\alpha - {}_nC_2 \cos^{n-3}\alpha \sin^3\alpha + -) \\ &= A + \iota B, \text{ say.}\end{aligned}$$

$\therefore$ $(\cos n\alpha, \sin n\alpha)$ and (A, B) are represented by the same vector, and $\quad\therefore\ \cos n\alpha = A,\ \sin n\alpha = B.$

[The last process, justified on p. 235, is called 'equating real and imaginary quantities on the sides of an equation'.]

Example. $\cos 4\alpha = \cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha$

$$= 8\cos^4\alpha - 8\cos^2\alpha + 1.$$

Evidently $\tan n\alpha = \dfrac{B}{A} = \dfrac{n\tan\alpha - {}_nC_3\tan^3\alpha +}{1 - {}_nC_2\tan^2\alpha + {}_nC_4\tan^4\alpha -}$.

In the equation $\sin n\alpha = B$, write $\alpha = \frac{\theta}{n}$ and rearrange, writing out the values of ${}_nC_2$, ${}_nC_4$, &c. Then $\sin\theta =$

$$\left(\cos\frac{\theta}{n}\right)^{n-1} \frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}} \cdot \theta - \left(\cos\frac{\theta}{n}\right)^{n-3} \left(\frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right)^3 \frac{\theta^3}{3\,!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots$$

$$\pm\left(\cos\frac{\theta}{n}\right)^{n-r}\left(\frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right)^r \frac{\theta^r}{r\,!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r-1}{n}\right) \mp$$

Increase n indefinitely, θ remaining finite.

Now $\underset{n\to\infty}{\mathrm{L}^t}\left(\frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right)^r = 1$, if r is finite (p. 103 and p. 105),

and $\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r-1}{n}\right) < 1, > 1 - \frac{r(r-1)}{n}$

(pp. 13 and 14).

$\therefore$ the limit of this product when r is finite is 1.

Also $1 > \left(\cos\frac{\theta}{n}\right)^n > \left(1-\frac{1}{2}\frac{\theta^2}{n^2}\right)^n$, when n is finite (p. 69, iii),

and $\underset{n\to\infty}{\mathrm{L}^t}\left(1-\frac{\theta^2}{2n^2}\right)^n = \underset{n\to\infty}{\mathrm{L}^t}\left\{\left(1-\frac{\theta^2}{2n^2}\right)^{n^2}\right\}^{\frac{1}{n}} = \mathrm{L}^t\left(e^{-\frac{\theta^2}{2}}\right)^{\frac{1}{n}}$

$$= e^0 = 1.$$

$$\therefore \quad \underset{n\to\infty}{\mathrm{L}^t}\left(\cos\frac{\theta}{n}\right)^n = 1.$$

These limits being combined, the terms in the expansion of $\sin\theta$, so far as θ^{2m+1}, where m is finite, may be written

$$\theta - \frac{\theta^3}{3\,!} + \frac{\theta^5}{5\,!} - \ldots - (-1)^m \cdot \frac{\theta^{2m-1}}{(2m-1)\,!}\,.$$

The remaining terms may be written

$R_m = (-1)^m \cdot \frac{\theta^{2m+1}}{(2m+1)\,!}$ multiplied by

$$\left\{1 - \frac{\left(1-\frac{2m+3}{n}\right)\left(1-\frac{2m+4}{n}\right)}{(2m+3)(2m+5)} \cdot \left(\frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right)^2 \cdot \frac{\theta^2}{\cos^2\frac{\theta}{n}} + \ldots\right\}.$$

This is convergent if $2m+3 > \theta \sec\frac{\theta}{n}$, and m can always be taken to satisfy this condition. Also $\frac{\theta^{2m+1}}{(2m+1)!}$ may be made as small as we please (p. 103).

Hence $\sin\theta = \underset{n\to\infty}{\text{Lt}}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots - (-1)^n \frac{\theta^{2n-1}}{(2n-1)!}\right)$.

By a similar argument it may be shown that

$$\cos\theta = \underset{n\to\infty}{\text{Lt}}\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots + (-1)^n \cdot \frac{\theta^{2n}}{(2n)!}\right).$$

If $\qquad z = \cos\theta + \iota\sin\theta, \quad \frac{1}{z} = \cos\theta - \iota\sin\theta,$

$$2\cos\theta = z + \frac{1}{z}, \qquad 2\sin\theta = -\iota\left(z - \frac{1}{z}\right),$$

and $\qquad z^t = \cos t\theta + \iota\sin t\theta, \; \frac{1}{z^t} = \cos t\theta - \iota\sin t\theta,$ where t is any integer.

$$2\cos t\theta = z^t + \frac{1}{z^t}, \qquad 2\sin t\theta = -\iota\left(z^t - \frac{1}{z^t}\right).$$

$$2^n\cos^n\theta = \left(z + \frac{1}{z}\right)^n$$

$$= z^n + \frac{1}{z^n} + n.\left(z^{n-2} + \frac{1}{z^{n-2}}\right) + {}_nC_2\left(z^{n-4} + \frac{1}{z^{n-4}}\right) + \ldots$$

$\therefore\; 2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + {}_nC_2\cos(n-4)\theta + \ldots.$

If n is even, the last term is ${}_nC_{\frac{n}{2}}$. If n is odd, the last term is ${}_nC_{\frac{n-1}{2}} \cdot \cos\theta$.

E.g. $4\cos^3\theta = \cos 3\theta + 3\cos\theta,$

$8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 6.$

EXAMPLE. Obtain similar expressions for $\sin^n\theta$, in terms of cosines of multiples of θ when n is even, and in terms of sines of multiples of θ when n is odd. Verify the results when $n = 2, 3, 4, 5$.

Series involving Complex Quantities.

We do not propose to deal with the convergency of series involving complex quantities, except in the simplest cases.

As with real quantities, a series is convergent if it has a unique finite limit when the number of its terms is increased indefinitely. l is said to be the limit of $f(z)$ for $z = z_1$, when the modulus of $f(z) \backsim l < \epsilon$, for all values of z whose modulus differs from that of z_1 by less than h, where h can be determined in terms of z_1 and ϵ, and ϵ is as small a quantity as we can assign. [Compare the definition on p. 101].

To explain this definition, let P, P_1, and Q represent z, z_1, and $f(z)$ respectively.

Describe a circle, radius h, about P_1. The modulus of every point within this circle is between $\rho \pm h$, where ρ is the modulus of z_1.

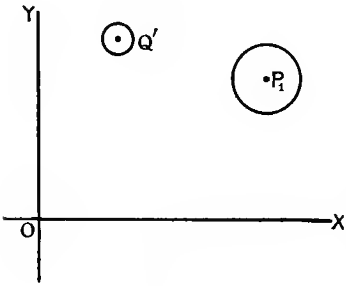


FIG. 93.

Let Q' represent l. Describe a circle, centre Q', radius ϵ.

The definition states that if P is within the circle round P_1, Q is within that round Q', and if by a suitable choice of h we can make ϵ as small as we please, then Q' is the limiting position of Q.

We may now add to the definition of a limit, that if $f(z)$ depends on any quantity, n, and is such that for an assigned value of z, $f(z)$ is always within a circle with known centre and assigned radius ϵ, when n is greater than some assigned value, then $f(z)$ has a unique limit when n is indefinitely increased. [Compare the latter part of the definition on p. 101, and the condition for convergency when $\underset{n \to \infty}{\mathrm{Lt}} R_n = 0$ on p. 107.]

THEOREM. *The series* $a_0 + a_1 z + a_2 z^2 + \ldots$ *is convergent, if* $a_0 + a_1 r + a_2 r^2 + \ldots$ *is convergent, where* r *is the modulus of* z.

Let $S_n = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1}$, and write R_n for

$$a_n z^n + a_{n+1} z^{n+1} + \ldots.$$

Let θ be the amplitude of z.

$$z^t = r^t (\cos t\theta + \iota \sin t\theta).$$

As on pp. 234–5, S_n can be expressed in the form $x' + \iota y'$.

$$R_n = r^n \{(a_n \cos n\theta + a_{n+1} r \cos \overline{n+1}\,\theta + \ldots)$$
$$+ \iota\,(a_n \sin n\theta + a_{n+1} r \sin \overline{n+1}\,\theta + \ldots)\}.$$

Write this $r^n(A+B\iota)$.

Then since no cosine or sine can be >1 or <1,

$$|A| \ngtr a_n + ra_{n+1} + \dots \text{ and } |B| \ngtr a_n + ra_{n+1} + \dots.$$

Since by hypothesis, $a_0 + a_1 r + \dots$ is convergent, $r^n A$ and $r^n B$ can each be made less than any assigned quantity by increasing n sufficiently.

Hence the modulus of R_n, viz. $r^n . \sqrt{A^2+B^2}$ can be made $<\epsilon$.

Let n have such a value.

If Q' represents S_n, then $S_n + R_n$ is always represented by a point within the circle, centre Q', radius ϵ.

Hence the series is convergent.

As with real quantities, we can evaluate a convergent series to any required degree of accuracy; that is, we can find a space on the diagram of complex quantities of as small area as we please, within which the point representing the limits of the series lies.

$1 + z + \frac{z^2}{2!} + \dots + \frac{z^t}{t!} + \dots$ is convergent for all values of z, since $1 + r + \frac{r^2}{2!} + \dots + \frac{r^t}{t!} + \dots$ is convergent (p. 109), where r is the modulus of z and real. Write $E(z)$, as before, for the limit of this series.

Example. The binomial series, general term $\frac{[n]_t}{t!} z^t$ is convergent if $-1 < r < 1$.

Multiplication of Convergent Series. [Compare pp. 113–14.]

Let $\qquad U_n = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n,$

and $\qquad V_n = b_0 + b_1 z + b_2 z^2 + \dots + b_n z^n,$

where all the coefficients are real positive quantities.

Let $W_n = a_0 b_0 + (a_0 b_1 + a_1 b_0) z + \dots$

$$+ (a_n b_0 + a_{n-1} b_1 + \dots a_0 b_n) z_n.$$

Let U'_n, V'_n, W'_n stand for the same series when r is written for z, corresponding to S_n, S'_n, Σ_n on p. 113.

Let U' and V' be convergent. Then W' is convergent and

$W' = U'V'$ as on p. 114; and U, V, and W are convergent, from pp. 242–3.

Now $W_{2n} - U_n V_n$ contains terms in $z^{n+1}, z^{n+2} \ldots z^{2n}$, all with positive coefficients, which also form part of the terms, all positive, of $W_{2n} - W_n$.

But since W is a convergent series the modulus of $W_{2n} - W_n$ may be made as small as we please by increasing n. *A fortiori* the modulus of $W_{2n} - U_n V_n$ may be made as small as we please. Hence $U_n V_n$ may be made to differ as little as we please from W_{2n} or from W_n. Hence in the limit $U \times V = W$.

Now work through the proof on p. 120 that

$$E(x_1) \times E(x_2) = E(x_1 + x_2),$$

writing z_1, z_2 for x_1, x_2. Every step will be found to apply, and the binomial theorem is only used with a positive integral index and will apply to complex quantities (compare p. 239).

Hence $\mathbf{E}(\mathbf{z}_1) \times \mathbf{E}(\mathbf{z}_2) = \mathbf{E}(\mathbf{z}_1 + \mathbf{z}_2)$ for all values of z.

We have here a close analogy with the first rule of indices (p. 2). In fact, if $z = x$, the equation last written is

$$e^{x_1} \times e^{x_2} = e^{x_1 + x_2}.$$

Now define e^z as $\underset{n \to \infty}{\mathrm{L}^t}\ 1 + z + \frac{1}{2}z^2 + \ldots + \frac{1}{n!} z^n = E(z)$,

and we have $$e^{z_1} \times e^{z_2} = e^{z_1 + z_2}.$$

As on pp. 2, 3, $\quad e^0 = 1, \quad e^{z_1} \times e^{z_2 - z_1} = e^{z_2}$,

$$\therefore\ e^{z_2} \div e^{z_1} = e^{z_2 - z_1}.$$

Also $e^{\frac{p}{q}z}$ is a q^{th} root of e^{pz}.

$(e^z)^m$ and e^{mz} are equal if m is integral, and have one value in common when z is fractional, whether m is positive or negative.

We may therefore use all the laws of indices in connexion with e^z.

[NOTE. No meaning has yet been given to a^z.]

Writing $z = x + y\iota$, take the case where $x = 0$, $y = \theta$.

$$e^{\theta\iota} = E(\theta\iota) = 1 + \theta\iota + \frac{(\theta\iota)^2}{2\,!} + \dots$$

$$= 1 + \theta\iota - \frac{\theta^2}{2\,!} - \iota\frac{\theta^3}{3\,!} + \frac{\theta^4}{4\,!} + - - + - - + \dots,$$

and $e^{-\theta\iota} = E(-\theta\iota)$

$$= 1 - \theta\iota - \frac{\theta^2}{2\,!} + \iota\frac{\theta^3}{3\,!} + \frac{\theta^4}{4\,!} - - + + - - + \dots.$$

Adding, $\dfrac{e^{\theta\iota} + e^{-\theta\iota}}{2} = \dfrac{E(\theta\iota) + E(-\theta\iota)}{2}$

$$= 1 - \frac{\theta^2}{2\,!} + \frac{\theta^4}{4\,!} - \dots = \cos\theta \text{ (p. 241).}$$

Subtracting and dividing by ι $\left(\text{the quantity } \left(0, \frac{\pi}{2}\right)\right)$,

$$\frac{e^{\theta\iota} - e^{-\theta\iota}}{2\iota} = \frac{E(\theta\iota) - E(-\theta\iota)}{2\iota}$$

$$= \theta - \frac{\theta^3}{3\,!} + \frac{\theta^5}{5\,!} - \dots = \sin\theta \text{ (p. 241).}$$

These real series are convergent for all values of θ.

The expressions $\dfrac{e^{\theta\iota} + e^{-\theta\iota}}{2}$ and $\dfrac{e^{\theta\iota} - e^{-\theta\iota}}{2\iota}$ are known as Euler's expressions for the cosine and sine.

Evidently
$$e^{\theta\iota} = E(\theta\iota) = \cos\theta + \iota\sin\theta,$$
$$e^{-\theta\iota} = E(-\theta\iota) = \cos\theta - \iota\sin\theta.$$
$$E(\pi\iota) = e^{\pi\iota} = -1 + \iota \times 0 = -1,$$
$$E\left(\frac{\pi}{2}\iota\right) = e^{\frac{\pi}{2}\iota} = \iota, \qquad E\left(-\frac{\pi}{2}\iota\right) = e^{-\frac{\pi}{2}\iota} = -\iota,$$
$$E\{(2k\pi + \theta)\iota\} = \cos(2k\pi + \theta) + \iota\sin(2k\pi + \theta)$$
$$= \cos\theta + \iota\sin\theta = E(\theta\iota),$$
$$E(\iota) = \cos 1 + \iota\sin 1 \text{ (where 1 means 1 radian),}$$
$$E(z) = E(x + y\iota) = E(x) \times E(y\iota) = e^x(\cos y + \iota\sin y),$$
and
$$E(x + \overline{y + 2k\pi}\,\iota) = E(x) \times E(\overline{y + 2k\pi}\,\iota)$$
$$= E(x) \times E(y\iota) = E(z).$$

$E(z)$ is therefore a periodic function, returning to the same value whenever 2π is added to the imaginary part of z.

[At this stage the student is advised to pause and reflect on the meaning of E, π, and ι, and to realize that $E(\pi\iota) = -1$ is a necessary result of the conventions under which the quantities are defined, and of the rules of operation which they have been defined as obeying, and to see that no inconsistency has been involved.]

EXAMPLES. Verify by Euler's expressions and the rules of indices that $\sin^2\theta + \cos^2\theta = 1$, $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = 1 - 2\sin^2\theta$,

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\,.\cos\frac{\alpha-\beta}{2}.$$

Trigonometrical Ratios of Complex Angles.

The idea of an angle and its ratios is by the preceding paragraphs separated from the first trigonometrical ideas.

We might in fact have defined $\sin x$ as the limit of $x - \frac{x^3}{3!} + \ldots$

and $\cos x$,, ,, $1 - \frac{x^2}{2!} + \ldots$

Now define $\sin z$, and $\cos z$, to which no meaning has yet been attached, thus:

$$\sin z = \mathrm{L}^t\left\{z - \frac{z^3}{3!} + \ldots\right\} = \frac{E(z\iota) - E(-z\iota)}{2\iota},$$

$$\cos z = \mathrm{L}^t\left\{1 - \frac{z^2}{2!} + \ldots\right\} = \frac{E(z\iota) + E(-z\iota)}{2\iota},$$

$$\tan z = \frac{\sin z}{\cos z} = \frac{1}{\cot z}; \quad \sec z = \frac{1}{\cos z}; \quad \operatorname{cosec} z = \frac{1}{\sin z}.$$

$$\sin^2 z + \cos^2 z = \frac{[E(z\iota) - E(-z\iota)]^2}{-4} + \frac{[E(z\iota) + E(-z\iota)]^2}{+4}$$

$$= \frac{-E(2z\iota) + 2E(0) - E(-2z\iota) + \;+\;+}{4}$$

$$= 1, \text{ since } E(0) = 1.$$

Similarly, or by using the index form, it can be shown that

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2,$$

and that all the formulae on pp. 57–8 are true for complex angles.

$$\cos(2k\pi + z) = \frac{E\{(2k\pi + z)\iota\} + E\{-(2k\pi + z)\iota\}}{2}$$

$$= \frac{E(2k\pi\iota)\times E(z\iota) + E(-2k\pi\iota)\times E(-z\iota)}{2}$$

$$= \frac{E(z\iota) + E(-z\iota)}{2} = \cos z.$$

Similarly for the other ratios.

The trigonometrical ratios of angles, real or complex, are periodic, returning to the same value whenever the real part is increased by 2π.

$\cos z$ is a function of z and can be separated into its real and imaginary parts and represented on the diagram of complex quantities thus:

$$\cos z = \frac{E(z\iota)+E(-z\iota)}{2} = \frac{E(\overline{x+y\iota}\,\iota)+E(-\overline{x+y\iota}\,\iota)}{2}$$

$$= \frac{E(x\iota-y)+E(-x\iota+y)}{2} = \frac{e^{-y}E(x\iota)+e^{y}E(-x\iota)}{2}$$

$$= \frac{e^{-y}(\cos x+\iota\sin x)+e^{y}(\cos x-\iota\sin x)}{2}$$

$$= \cos x\cdot\frac{e^{y}+e^{-y}}{2}+\iota\sin x\cdot\frac{e^{y}-e^{-y}}{2}.$$

E.g. If $z=\left(\frac{\pi}{4},\ 1\right)$ (P),

then $\cos z$

$$= \frac{1}{\sqrt{2}}\left\{\frac{e^{1}+e^{-1}}{2}+\iota\frac{e^{1}-e^{-1}}{2}\right\}$$

$= (1{\cdot}09,\ {\cdot}83)$ (Q).

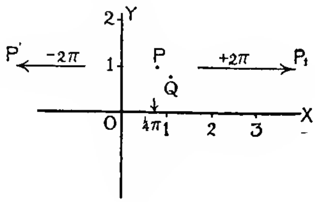


FIG. 94.

EXAMPLE. Express the other ratios similarly.

If $\cos z = u$ (Q), i.e. $z=\cos^{-1}u$, where u is known, P may have any of the positions ... P'', P', P, P_1, P_2 ....

EXAMPLE. Solve completely on the graph the equation

$$\sin z = 2+3\iota.$$

Hyperbolic Functions.

DEFINITION. $\dfrac{E(z)+E(-z)}{2} = \dfrac{e^{z}+e^{-z}}{2} = \mathrm{L}^{t}\left(1+\dfrac{z^2}{\underline{|2}}+\dfrac{z^4}{\underline{|4}}+\dots\right)$

is called the hyperbolic cosine of z and written $\cosh z$.

$\dfrac{e^{z}-e^{-z}}{2} = \mathrm{L}^{t}\left(z+\dfrac{z^3}{\underline{|3}}+\dots\right)$ is called the hyperbolic sine of z and written $\sinh z$.

tanh z is defined as $\dfrac{\sinh z}{\cosh z}$, &c.

Then $\cosh^2 z - \sinh^2 z = \frac{1}{4}\{E(z)+E(-z)\}^2$
$-\frac{1}{4}\{E(z)-E(-z)\}^2 = 1.$

If z is real these expressions are all real.

$$\cosh^2 x - \sinh^2 x = 1.$$

[If, in co-ordinate geometry, $X = a\cosh\theta$, and $Y = b\sinh\theta$, $\dfrac{X^2}{a^2} - \dfrac{Y^2}{b^2} = 1$, and the locus of X, Y is a hyperbola; hence the name.]

EXAMPLES. Show that the hyperbolic functions have an imaginary period.

Show that $\cosh z = \cos y \cosh x + \iota \sin y \sinh x$.

Given that $\cosh z = 2 + 3\iota$, show that $z = 1{\cdot}98 + \iota(2k\pi + 1{\cdot}00\ldots)$.

Show that $\cosh^{-1} u = \log(u \pm \sqrt{u^2-1})$ when u is real.

These functions, ordinary or hyperbolic, having real or imaginary periods, are of great importance in developing the theory of wave motion.

The inverse hyperbolic functions ($\sinh^{-1} x$, &c.) are of use in integration.

EXAMPLE. Show that

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C$$
$$= \log_e(x + \sqrt{x^2-a^2}) + C.$$

The graph of $\cosh x$ can be obtained as follows:

x	e^x	e^{-x}	$\cosh x$
0	1	1	1
1	2·72	·37	1·54
2	7·39	·14	3·76
3	20·08	·05	10·1

Thus, if $u = e^2$,

$$2 = \log_e u$$
$$\log_{10} u = 2 \times \log_{10} e = {\cdot}8686$$
$$u = 7{\cdot}39.$$

Also $\cosh(-x) = \cosh x$.

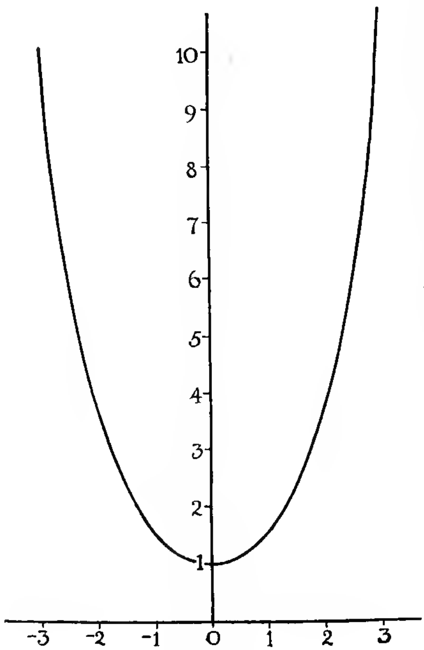


FIG. 95.

This curve is the *catenary*, the form taken by a uniform chain suspended from two points.

EXAMPLES. Draw the graph of $\sinh x$, and of $\tanh x$, and from them find $\sinh^{-1} 2$, $\tanh^{-1} 4$.

Show that $\sinh z = -\iota \sin z\iota$, $\quad \cosh z = \cos z\iota$,

$\sin z\iota = \iota \sinh z$, $\quad \tanh z = -\iota \tan z\iota$,

$\tan z\iota = \iota \tanh z$.

The following notes show how the theory of complex quantities is further developed. The student is referred to Hobson's *Plane Trigonometry*, pp. 282 seq.

u is defined as the logarithm of z, if u has any value that satisfies $z = E(u)$. Then u is written as $\text{Log}_e z$.

a^z is defined as $E(v)$ where $v = z \text{ Log}_e a$ (compare p. 123).

If a is real a^z has a meaning at once. If a is complex, $\text{Log}_e a$ is to be interpreted as just above.

a^z is shown, with certain restrictions, to conform to the laws of indices.

[If z is real but incommensurable, we can give a similar meaning thus:

$$4^{\sqrt{3}} = 1 + \sqrt{3} \,.\, \log_e 4 + \frac{1}{2\,!}(\sqrt{3} \,.\, \log_e 4)^2 + \dots .]$$

The limit of $1 + nz + \dfrac{n(n-1)}{1 \,.\, 2} z^2 + \dots$ is shown to be a value of $(1+z)^n$. (Binomial Theorem.)

$\log_e(1+z)$ is shown to be the limit of $z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \dots$, when the modulus of z is not greater than 1.

Logarithms are found to be functions with an imaginary period.

As an example of the use to which these definitions and theorems can be put, the following incomplete proof is outlined:

$$e^{\theta\iota} = \cos\theta + \iota \sin\theta,$$

$$\theta\iota = \log_e(\cos\theta + \iota\sin\theta) = \log_e \cos\theta \times \log_e(1 + \iota\tan\theta).$$

$$\therefore \theta\iota = \text{imaginary part of } \iota\tan\theta - \tfrac{1}{2}(\iota\tan\theta)^2 + \tfrac{1}{3}(\iota\tan\theta)^3 - \dots$$

If $\tan\theta = x$ we have one value of

$$\tan^{-1} x = \theta = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \dots .$$

It can be shown that if $0 < x \not> \frac{1}{4}\pi$, the series gives the acute angle.

In particular, if $\theta = \frac{1}{4}\pi$, $\tan\theta = 1 = x$,

and $\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots\right)$. Gregory's series.

There is no way of obtaining this, which in a slightly developed form gives the easiest way of evaluating π, without all the definitions and proofs indicated, except by a troublesome method of limits or by integrating an infinite series, which requires a considerable development of theory.

With this practical outcome of the very abstract theory of complex quantities we may conclude this section.

SECTION IX

CO-ORDINATE GEOMETRY IN THREE DIMENSIONS

THE methods of analytical geometry can readily be extended to three dimensions.

The Point.

Let XOY, YOZ, ZOX be three planes mutually at right angles (planes of reference), intersecting in the axes OX, OY, OZ. For convenience of drawing consider OZ as vertical.

The position of P, any point in space, is determined, when its perpendicular distances $PK\,(x)$, $PL\,(y)$, and $PM\,(z)$ from the planes YZ, ZX, and XY respectively are known.

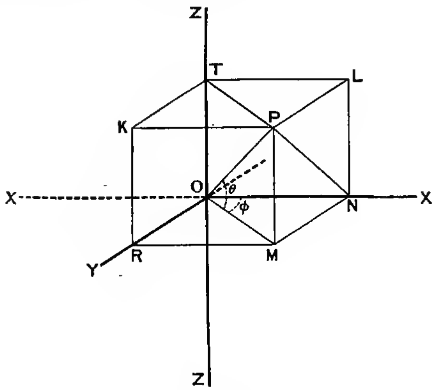


FIG. 96.

Complete the rectangular solid as in Figure 96. Then ON is at right angles to every line in the plane PMN, and $\therefore$ ONP is a right angle. Let $OP = r$, and let $l = \cos NOP$, $m = \cos ROP$, $n = \cos TOP$.

ON, OR, OT are the projections of OP on the axes.

$\therefore$ $x = rl$, $y = rm$, $z = rn$.

But $r^2 = ON^2 + NM^2 + MP^2 = x^2 + y^2 + z^2 = r^2(l^2 + m^2 + n^2)$.

$$\therefore \mathbf{l^2 + m^2 + n^2 = 1}. \quad \ldots \quad \text{(i)}$$

l, m, n are called the *direction cosines* of the direction OP.

The point may also be determined in spherical polar co-ordinates thus: Let a plane revolve round OZ from the initial position XOZ through an angle ϕ to the position MOZ, and let a radius revolve in the plane MOZ from OM through an angle θ to the position OP, and take a distance r on OP. Then r, ϕ, θ give the point. Here

$$x = r\cos\theta\cos\phi, \quad y = r\cos\theta\sin\phi, \quad z = r\sin\theta.$$

ϕ and θ correspond to longitude and latitude.

Let P, Q be the points $(x_1 y_1 z_1)$, $(x_2 y_2 z_2)$, and let $PQ = d$.

Then the projection of PQ on any line equals the difference of the projections of OP, OQ. Hence the projections of PQ on the axes are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$,

and $$\therefore \mathbf{d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \quad \ldots \quad \text{(ii)}$$

Compare p. 131 (A).

The co-ordinates of a point dividing PQ in the ratio $m:n$ are of the same form as (B), p. 132.

The Plane.

A plane is determined, if the perpendicular on it from O is known, in magnitude (p) and direction $(l\ m\ n)$.

Let D be the foot of this perpendicular and $P\ (x\ y\ z)$ any point on the plane.

Then (using the letters of Figure 96) the projection of OP on OD equals the sum of the projections of ON, NM, MP on OD.

But the projection of OP is p, since OD is perpendicular to DP, and l, m, n are the cosines of the angles between ON, NM, MP, and OD.

$\therefore$ the equation, $$\mathbf{lx + my + nz = p}, \quad \ldots \quad \text{(iii)}$$

is satisfied by every point on the plane, and is the equation of the plane.

The general equation of the first degree in $(x\ y\ z)$, viz. $ax + by + cz + d = 0$, can be put into the form of (iii) by taking

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = -\frac{p}{d} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}} = \frac{+1}{+\sqrt{a^2+b^2+c^2}} \quad \text{(iv)}$$

We make the convention that the positive root of the surd is to be taken.

This general equation therefore represents a plane; the direction cosines of the perpendicular to it are in the ratios $a:b:c$, and the perpendicular from the origin to it is

$$-d \div \sqrt{(a^2+b^2+c^2)}.$$

If Q is any point $(x_1 y_1 z_1)$ whose distance from the plane measured in the direction $(l\ m\ n)$ is p_1, then $lx_1+my_1+nz_1$ = projection of OQ on $OD = p-p_1$.

$$\therefore\ \mathbf{p_1 = p-(lx_1+my_1+nz_1) = -\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}}. \qquad \text{(v)}$$

Compare p. 138 (H), and p. 140. In formula (v) the origin is taken as being on the negative side of the plane if d is positive, and vice versa.

Evidently $x = k$ represents a plane parallel to ZOY, &c.

The Straight Line, and Angles.

From $P\,(x_1\,y_1\,z_1)$ let a line be drawn in the direction $(l\ m\ n)$, let $Q\,(x\ y\ z)$ be any point on the line, and let $PQ = r$.

Then $x-x_1 = rl$, &c., and the equations

$$\mathbf{\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}} \quad . \quad . \quad . \quad . \qquad \text{(vi)}$$

determine the locus of Q, and are therefore the equations of a straight line. Compare p. 136 (E) and p. 185.

The equations of the *line joining two points* $(x_1 y_1 z_1)$ and $(x_2 y_2 z_2)$ may be written as in (F), p. 137,

$$\mathbf{\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}} \quad . \quad . \quad . \quad . \quad . \qquad \text{(vii)}$$

The *angle between two intersecting lines* whose directions are $(l\,m\,n)$ and $(l'\,m'\,n')$ is thus obtained:

Let OP, OP' be lines through the origin parallel to the given lines, each of unit length, so that $(l\,m\,n)$, $(l'\,m'\,n')$ are the co-ordinates of P, P'.

Using equations (ii), p. 53, and (i) and (ii), p. 252, we have

$$\cos \mathbf{POP'} = (OP^2 + OP'^2 - PP'^2) \div 2\, OP \,.\, OP'$$

$$= \tfrac{1}{2}\{1 + 1 - (l' - l)^2 - (m' - m)^2 - (n' - n)^2\}$$

$$= \tfrac{1}{2}\{2 - 1 - 1 + 2(ll' + mm' + nn')\} = \mathbf{ll' + mm' + nn'}. \quad \text{(viii)}$$

The *angle between two planes* equals the angle between the perpendiculars to them from the origin, and may be written

$$\cos^{-1}(ll' + mm' + nn')$$

$$= \cos^{-1}\frac{\mathbf{aa' + bb' + cc'}}{\sqrt{\{(\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2)(\mathbf{a}'^2 + \mathbf{b}'^2 + \mathbf{c}'^2)\}}}, \quad . \quad . \quad \text{(ix)}$$

where $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ are the planes.

The *planes* are therefore at *right angles* to each other, if

$$\mathbf{aa' + bb' + cc' = 0}. \quad \text{(Compare p. 136 (D).)} \quad . \quad . \quad \text{(x)}$$

They are *parallel* if $\dfrac{\mathbf{a}}{\mathbf{a}'} = \dfrac{\mathbf{b}}{\mathbf{b}'} = \dfrac{\mathbf{c}}{\mathbf{c}'}$. (Compare p. 136 (C).) (xi)

If $(l_1\, m_1\, n_1)$ is the direction of the *line of intersection of two planes*, it is at right angles to the normal to each.

If $(l\, m\, n)$ is the direction of one of the normals

$$l_1 l + m_1 m + n_1 n = 0 \text{ by (viii)},$$

and $\therefore\ l_1 a + m_1 b + n_1 c = 0$ by (iv). Similarly

$$l_1 a' + m_1 b' + n_1 c' = 0.$$

$$\therefore\ \mathbf{l_1 : m_1 : n_1 = bc' - b'c : ca' - c'a : ab' - b'a}. \quad . \quad . \quad \text{(xii)}$$

Surfaces.

An equation connecting x, y, and z represents in general a surface.* For let $f(xyz) = 0$. Then, if we consider a particular value, k, of z, we obtain $f(xyk) = 0$ as the intersection of the locus with the plane $z = k$. Thus, as we vary k, we get a succession of two-dimensional loci.

This conception is readily visualized by considering these (xy) loci as horizontal contour lines of a surface. The equation $x^2 - 4y^2 = 8z$ is represented in Figure 97 as in a contour map,

* The exceptions are when there are no real values of x, y, and z satisfying the equation, in which case the surface is imaginary, and when the surface degenerates into lines or points in limiting cases.

showing the projections of horizontal sections on the plane $z = 0$, between the limits $x = \pm 9$, $y = \pm 5$.

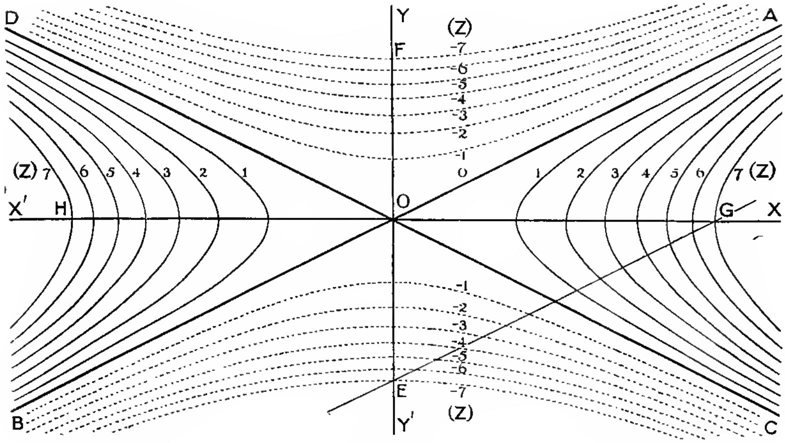


FIG. 97.

The equations of the contours $z = \ldots 4, 3, 2, 1, 0, -1, -2, -3, -4 \ldots$ are

$$\ldots, x^2 - 4y^2 = 32, \ldots, x^2 - 4y^2 = 0, \ldots, x^2 - 4y^2 = -32, \ldots,$$

and the contours are a family of hyperbolas with their conjugates, having the asymptotes $x = \pm 2y$.

The surface is of the form of a mountain pass. The ascent from E is at first precipitous, but becomes easier as the col O is approached. The vertical section by $x = 0$ is the parabola $-y^2 = 2z$; the symmetrical route from E to F is a parabola turned downwards with vertex at O. The vertical section by $y = 0$ is the parabola $x^2 = 8z$, turned upwards, vertex also at O; as we take breath at the col and look left and right, we see that we are at the bottom of an infinite U-shaped figure, GOH. Instantaneously we are on a level plane whose trace we can tell by the straight level paths AB, CD. It will presently be shown (pp. 267–8) that every vertical section parallel to AB or to CD is a straight line, which is the steeper the further it is from O. EG, drawn in the figure, rises from the valley up the mountain at a gradient about $\tan 59°$. As we look forward or backward from O we see infinite valleys whose horizontal sections are hyperbolas. As we descend and look left and right we see that we are always at the vertex of a parabola of unchanging size ($x^2 = 8z$).

EXAMPLES.

Trace the contours of

(1) The four surfaces $\frac{x^2}{9} \pm \frac{y^2}{16} \pm \frac{z^2}{25} = 1$;

(2) „ three „ $\frac{x^2}{9} \pm \frac{y^2}{16} \pm \frac{z^2}{25} = 0$, omitting $+\ +$;

(3) $x^2 + 4y^2 = 8z$;

(4) $x^2 = 4y + 8z$;

(5) The surface $4x^2 + 9y^2 + 20xy + \log_{10} z = 0$ from $z = \cdot 1$ to $z = 1$.

(1) gives the central conicoids, (2) the cones, (3) a paraboloid, (4) a cylinder (see pp. 260–2), and (5) is called the correlation surface.

Transformation of Co-ordinates.

To transfer the origin to $(x_1 y_1 z_1)$, the directions of the axes being unchanged, write $x + x_1$, $y + y_1$, $z + z_1$ for x, y, z as on p. 142.

Rotation of rectangular axes, origin unchanged. Let the direction cosines of the new axes, OX_1, OY_1, OZ_1, referred to the old, be $(l_1 m_1 n_1)$, $(l_2 m_2 n_2)$, $(l_3 m_3 n_3)$.

Let the co-ordinates of a point, P, be $(x\, y\, z)$ referred to the old axes and $(x'\, y'\, z')$ referred to the new. Using the same letters for the old axes as in Figure 96, and corresponding letters for the new, we may write $ON' = x'$, $N'M' = y'$, and $M'P' = z'$.

The projection of OP equals the sum of the projections of ON', $N'M'$, and $M'P'$.

Projecting on OX, OY, OZ in succession, we have

$$\mathbf{x} = \mathbf{l_1 x' + l_2 y' + l_3 z'}, \quad \mathbf{y} = \mathbf{m_1 x' + m_2 y' + m_3 z'},$$

$$\mathbf{z} = \mathbf{n_1 x' + n_2 y' + n_3 z'}. \quad \ldots \ldots \quad \text{(xiii)}$$

This substitution effects the required transformation of co-ordinates.

The nine direction cosines are connected by the six equations

$$1 = l_1^2 + m_1^2 + n_1^2 = l_2^2 + m_2^2 + n_2^2 = l_3^2 + m_3^2 + n_3^2$$

by equation (i),

and $\quad 0 = l_1 l_2 + m_1 m_2 + n_1 n_2 = l_2 l_3 + m_2 m_3 + n_2 n_3$

$$= l_3 l_1 + m_3 m_1 + n_3 n_1 \quad \text{(x) and (iv).}$$

A function remains of the same degree after transformation, for (xiii) shows that its degree cannot be raised; nor can it be lowered, for on transformation back it would then be raised.

Since l_1, l_2, l_3 are the direction cosines of OX referred to OX_1, OY_1, OZ_1, &c., it follows that

$$1 = l_1^2 + l_2^2 + l_3^2 = m_1^2 + m_2^2 + m_3^2 = n_1^2 + n_2^2 + n_3^2,$$

and $\quad 0 = l_1 m_1 + l_2 m_2 + l_3 m_3 = m_1 n_1 + m_2 n_2 + m_3 n_3$

$$= n_1 l_1 + n_2 l_2 + n_3 l_3,$$

equations which can be shown to be algebraically equivalent to the former six.

EXAMPLE. Obtain the formulae of p. 174 by putting

$$l_3 = m_3 = 0.$$

The General Equation of the Second Degree.

This may be written

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0. \quad \text{(xiv)}$$

The centre. As on page 176 refer to a new origin $(\bar{x}\,\bar{y}\,\bar{z})$, O', chosen so that the new coefficients of x, y, z shall be zero.

We must have

$$\left.\begin{aligned} a\bar{x} + h\bar{y} + g\bar{z} + u &= 0 \\ h\bar{x} + b\bar{y} + f\bar{z} + v &= 0 \\ g\bar{x} + f\bar{y} + c\bar{z} + w &= 0 \end{aligned}\right\} \quad \text{(xv)}$$

The solution of these equations, in the notation of p. 175, is found to be

$$\frac{\bar{x}}{uA + vH + wG} = \frac{\bar{y}}{uH + vB + wF} = \frac{\bar{z}}{uG + vF + wC} = -\frac{1}{\Delta},$$

and is always possible unless $\Delta = 0$.

After transformation (xiv) becomes

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+d'=0, \quad . \quad \text{(xvi)}$$

where $d'=d-(Au^2+Bv^2+Cw^2+2Fvw+2Gwu+2Huv)\div\Delta$.

If a point $P(x_1 y_1 z_1)$ is on (xvi), then $P'(-x_1-y_1-z_1)$ is also on it, and PP' is bisected at O'. O' is the centre of the surface.

The principal planes. We shall now show that the terms in yz, zx, xy can be removed by rotation of the axes of reference. The method of p. 177 becomes unworkable and we must proceed indirectly.

Let $\mathbf{F}=(a-\lambda)x^2+(b-\lambda)y^2+(c-\lambda)z^2+2fyz+2gzx+2hxy$.

Choose an angle θ to satisfy the equation $\tan 2\theta=\dfrac{2h}{a-b}$.

Then by methods analogous to those of pp. 176–8, it can be shown that $\mathbf{F}$ is identically equal to

$$a_1\{x\cos\theta+y\sin\theta-(g'\cos\theta+f'\sin\theta)z\}^2$$
$$+b_1\{x\sin\theta-y\cos\theta+(f'\cos\theta-g'\sin\theta)z\}^2,$$

if $(a-\lambda)(b-\lambda)(c-\lambda)+2fgh-(a-\lambda)f^2-(b-\lambda)g^2$

$$-(c-\lambda)h^2=0, \quad . \quad \text{(xvii)}$$

where $a_1+b_1=a+b-2\lambda$, $a_1b_1=(a-\lambda)(b-\lambda)-h^2$, $Cg'=G$, $Cf'=F$, $C=a_1b_1$, $G=hf-(b-\lambda)g$, $F=gh-(a-\lambda)f$.

Equation (xvii), called the *discriminating cubic,* has always at least one real root (p. 239), say λ_1.

Hence if p and p' are written for the distances of a point from the planes $\quad x\cos\theta+y\sin\theta-(g'\cos\theta+f'\sin\theta)z=0$

and $\quad x\sin\theta-y\cos\theta+(f'\cos\theta-g'\sin\theta)z=0$,

the locus $\mathbf{F}=0$, when $\lambda=\lambda_1$, becomes by (v), p. 253,

$$a'p^2+b'p'^2=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xviii)}$$

where a', b' depend on a_1, b_1, f', g', and θ and are constants.

Take the line of intersection of these planes as the new axis of z, and let their equations referred to new axes of x and y be $x\cos\alpha+y\sin\alpha=0$, $x\cos\beta+y\sin\beta=0$.

The locus $\mathbf{F} = 0$ becomes

$$a' (x \cos \alpha + y \sin \alpha)^2 + b' (x \cos \beta + y \sin \beta)^2 = 0.$$

Now by p. 177 this can always be transformed to the form $a'_1 x^2 + b'_1 y^2 = 0$, by rotation of the axes of x and y without affecting that of z.

This is independent of z, so that if a point $(x_1 y_1)$ satisfy the equation, then $(x_1 y_1 z)$ is on $\mathbf{F} = 0$ for all values of z. Hence if we take the value $z = z_1$, the eight points $(\pm x_1 \pm y_1 \pm z_1)$ are on the locus, which is therefore symmetrical with regard to the three new planes of reference.

Hence $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy - \lambda_1 (x^2 + y^2 + z^2)$ can be transformed into a form without terms in yz, zx, xy. But $x^2 + y^2 + z^2 = OP^2 = x'^2 + y'^2 + z'^2$, where P is $(x\, y\, z)$ before and $(x'\, y'\, z')$ after transference.

$\therefore$ the product terms can be removed from the first part of the expression alone, and equation (xvi) becomes of the form

$$a_1 x^2 + b_1 y^2 + c_1 z^2 + d' = 0,$$

which is $\quad a_1 x^2 + b_1 y^2 + c_1 z^2 = 0$, if $d' = 0$, . . . (xix)

and, if $d' \neq 0$, $\quad \mathbf{A}x^2 + \mathbf{B}y^2 + \mathbf{C}z^2 = 1$, ... (xx)

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are written for $-\dfrac{a_1}{d'}$, $-\dfrac{b_1}{d'}$, $-\dfrac{c_1}{d'}$.

The new planes of reference are called the *principal planes* of the surface.

The discriminating cubic may be written

$$\lambda^3 - \lambda^2 (a + b + c) + \lambda (ab + bc + ca - f^2 - g^2 - h^2) - \Delta = 0.$$

It can be shown that its roots are a_1, b_1, c_1, that is, are $-d'\mathbf{A}$, $-d'\mathbf{B}$, $-d'\mathbf{C}$. For the preceding analysis shows that $\mathbf{F} = 0$ can be transformed into the geometric statement (xviii) only if λ is a root of (xvii), and that then the result is

$$(a_1 - \lambda)\, x^2 + (b_1 - \lambda)\, y^2 + (c_1 - \lambda)\, z^2 = 0,$$

which must also represent the geometric condition; but if $\lambda = a_1$, the equation becomes $(b_1 - a_1) y^2 + (c_1 - a_1) z^2 = 0$, that is (perpendicular from $ZOX)^2 \times$ const. + (perpendicular from $XOY)^2 \times$ const. $= 0$, the same form as (xviii). Hence a_1 and (by similar reasoning) b_1 and c_1 are the three roots of (xvii).

The equation of p. 178, line 10, is analogous in two dimensions.

If $\Delta = 0$, one root of (xvii) is zero. Thus in the case where we cannot transfer to the centre, the removal of the product terms gives the form

$$a_1 x^2 + b_1 y^2 + c_1 z^2 + 2u'x + 2v'y + 2w'z + d_1 = 0,$$

where a_1, b_1, or c_1 is zero.

Take $c_1 = 0$ and transfer to the origin

$$\left(\frac{-u'}{a_1}, \frac{-v'}{b_1}, \frac{-d_1}{2w'} + \frac{b_1 u'^2 + a_1 v'^2}{2w' a_1 b_1}\right).$$

The equation becomes $a_1 x^2 + b_1 y^2 + 2w'z = 0$. . . . (xxi)

If $w' = 0$, we have the form $a_1 x^2 + b_1 y^2 = \frac{u'^2}{a_1} + \frac{v'^2}{b_1} - d_1$. (xxii)

If a second root of (xvii) (say b_1) also is zero, we get after a simple transference

$$a_1 x^2 + 2v'y + 2w'z = 0. \quad . \quad . \quad . \quad . \quad \text{(xxiii)}$$

Classification of Conicoids.

All forms of the locus represented by the general equation (xiv) are called *conicoids*. In all cases (xiv) can be simplified into one or other of the forms (xix) to (xxiii). Of these (xx) is the general form and the others are limiting or special cases.

(xix) is a cone* with the origin as vertex, since if any point on the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ lies on it, the whole line lies on it. If a_1, b_1, c_1 are of the same sign the surface reduces to one point, the origin. If $a_1 = 0$, and b_1, c_1 are of different signs, it becomes the planes $y = \pm \sqrt{-\frac{c_1}{b_1}} \cdot z$.

(xxi) is drawn in the case when a_1 and b_1 are of different signs in Fig. 97, and is called an *hyperbolic paraboloid*. If a_1 and b_1 are of the same sign, the surface is entirely above (w' negative) or entirely below (w' positive) the plane $z = 0$, and all horizontal sections are similar ellipses; this is an *elliptic paraboloid*.

* A *cone* is a surface generated by a line, which always passes through a fixed point (the vertex) and intersects a given plane in any assigned curve.

(xxii) and (xxiii) are cylinders.* In (xxii) the generating line is vertical, in (xxiii) it is in the direction $(0, w', -v')$.

All conicoids except the paraboloids and (xxiii) have three planes of symmetry, viz. the planes of reference in (xix), (xx), (xxi), and (xxii).

Form (xx) gives the central conicoids, the origin being the centre.

If **A**, **B**, **C** are all positive, the surface is called an *ellipsoid*. If, further, two of these (**A**, **B**) are equal, it is a *spheroid*, *prolate* if $\mathbf{A} = \mathbf{B} > \mathbf{C}$, *oblate* if $\mathbf{A} = \mathbf{B} < \mathbf{C}$. If $\mathbf{A} = \mathbf{B} = \mathbf{C}$ it is a *sphere*.

The ellipsoid may be written $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$, and is a closed surface, contained within the rectangular box $x = \pm\alpha$, $y = \pm\beta$, $z = \pm\gamma$. α, β, γ are its semi-axes. Its volume is $\frac{4}{3}\pi\alpha\beta\gamma$.† Sections parallel to each principal plane are similar ellipses.

If **A**, **B**, **C** are all negative, the surface is imaginary.

If of **A**, **B**, **C** two are positive and the third (**C**) negative, the equation may be written $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} - \frac{z^2}{\gamma^2} = 1$. This is called an *hyperboloid of one sheet*. All horizontal sections are similar ellipses, all sections parallel to YOZ are similar hyperbolas, as are those parallel to XOZ. It is readily shown that the cone $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} - \frac{z^2}{\gamma^2} = 0$ lies within it in finite regions, while all its generating lines are asymptotes to vertical sections. The shape of the surface is that of an infinite dice-box. α and β are real semi-axes.

If one (**A**) is positive and the others negative, the equation may be written $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} - \frac{z^2}{\gamma^2} = 1$. This is called an *hyperboloid of two sheets*. The surface touches and lies beyond the planes

* A *cylinder* is a surface generated by a line, which moves parallel to itself and intersects a given plane in any assigned curve.

† The area of the ellipse on the plane $z = z_1$ is $\pi\alpha\beta\left(1 - \frac{z_1^2}{\gamma^2}\right)$, p. 169. The volume is $2\int_0^\gamma \pi\alpha\beta\left(1 - \frac{z^2}{\gamma^2}\right).dz$. Compare p. 215.

$x = \pm\alpha$. All sections parallel to YOZ are similar ellipses. All sections parallel to XOY are similar hyperbolas, as are sections parallel to XOZ. The generating lines of the cone

$$\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} - \frac{z^2}{\gamma^2} = 0$$

are asymptotes, and the surface lies within this cone. α is a real semi-axis.

The shapes of all these curves can be examined by the method of p. 256, with the help of the examples there set.

The paraboloids may be obtained from the ellipsoid or hyperboloids by transferring to a vertex and following the method of pp. 155–6.

If of **A**, **B**, **C** two are equal, say **A** = **B**, all horizontal sections (in xix, xx, xxi, xxii) are circles, and the surface can be obtained by revolving the curve $\mathbf{A}v^2 + \mathbf{C}z^2 = 0$ or 1, or $-2w'z$, or constant, about OZ. Such surfaces are *surfaces of revolution*.

[If a reflector is made in the shape of a paraboloid of revolution, every ray of light emanating from the focus of its parabolic sections is reflected parallel to the axis (see p. 173). For an ellipsoid of revolution the rays return to the other focus.]

Intersections of the Equations of the 1st and 2nd degrees.

Let $lx + my + nz = p$ be any plane. Transfer this and the general equation of the second degree, so that $lx + my + nz = 0$ becomes the plane X_1OY_1. The plane becomes $z = p$ and the general equation remains in the general form (xiv) with its coefficients changed, say, to $a', \ldots f', \ldots u', \ldots$.

The section on the plane $z = p$ is

$$a'x^2 + b'y^2 + c'p^2 + 2f'yp + 2g'px + 2h'xy$$
$$+ 2u'x + 2v'y + 2w'p + d = 0,$$

that is, a conic section.

As p varies, the resulting sections are similar to each other, as their shape depends only on a', b', h'.

The centre of any section is given by

$$a'x + h'y + g'p + u' = 0 = h'x + b'y + f'p + v' \text{ and } z = p,$$

and as p varies this is the fixed line through the centre of the conicoid given by the first two of equations (xv) with a', &c. written for a, &c.

Hence every plane section of every conicoid is a conic section, and parallel planes give similar conics whose centres are on a diameter of the conicoid.

The Line and the Conicoid.

Let $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$ be any line and

$$\mathbf{A}x^2 + \mathbf{B}y^2 + \mathbf{C}z^2 = 1$$

be any central conicoid, where $(x_1 y_1 z_1)$ is J. Let P_1, P_2 be points of intersection of the line and surface. Then, as on p. 185, $r_1 (=JP_1)$ and $r_2 (= JP_2)$ are roots of

$$\mathbf{A}(x_1 + rl)^2 + \mathbf{B}(y_1 + rm)^2 + \mathbf{C}(z_1 + rn)^2 = 1,$$

i.e. of $r^2 (\mathbf{A}l^2 + \mathbf{B}m^2 + \mathbf{C}n^2) + 2r(\mathbf{A}lx_1 + \mathbf{B}my_1 + \mathbf{C}nz_1)$

$$+ \mathbf{A}x_1{}^2 + \mathbf{B}y_1{}^2 + \mathbf{C}z_1{}^2 - 1 = 0. \quad . \quad . \quad . \quad \text{(xxiv)}$$

The Tangent Plane.

Let J move up to P_1, then $\mathbf{A}x_1{}^2 + \mathbf{B}y_1{}^2 + \mathbf{C}z_1{}^2 - 1 = 0$ and r_1 is zero.

The plane $\mathbf{A}\mathrm{x}x_1 + \mathbf{B}\mathrm{y}y_1 + \mathbf{C}\mathrm{z}z_1 = 1 \quad . \quad . \quad . \quad . \quad$ (xxv)

then passes through P_1; it contains the line JP_1P_2 if its normal is perpendicular to JP_1P_2, i.e. if $\mathbf{A}x_1 l + \mathbf{B}y_1 m + \mathbf{C}z_1 n = 0$ (from equations (iv) and (viii)).

But this is the condition that the second root of (xxiv) is zero. Hence every line in (xxv) through P_1 meets the surface in two coincident points, and ∴ (xxv) is the equation of the plane which touches the surface at $(x_1 y_1 z_1)$. [Compare the equation $\mathbf{A}xx_1 + \mathbf{B}yy_1 = 1$ (p. 158).]

It now easily follows that $lx + my + nz = p$ touches the surface if $p^2 = \frac{l^2}{\mathbf{A}} + \frac{m^2}{\mathbf{B}} + \frac{n^2}{\mathbf{C}}$.

Pole and Polar.

The tangent plane at $(x'y'z')$ passes through a point $(\xi\eta\zeta)$, Q, if $\mathbf{A}\xi x' + \mathbf{B}\eta y' + \mathbf{C}\zeta z' = 1$. Hence $\mathbf{A}x\xi + \mathbf{B}y\eta + \mathbf{C}z\zeta = 1$ is a plane containing the points of contact of all tangent planes through Q, and may be called the plane of contact for Q.

Then, as on p. 163, the tangent planes at the points, where a varying plane through a fixed point $P\ (x_1 y_1 z_1)$ cuts the surface, are concurrent at a point (Q) whose locus is

$$\mathbf{A}xx_1 + \mathbf{B}yy_1 + \mathbf{C}zz_1 = 1.$$

This plane is called the *polar* of P, and P is the *pole* of the plane.

If Q lies on the polar of P, P lies on the polar of Q.

Notice from equation (xxiv) that, just as on p. 185, if lines JPP', JQQ' are drawn in fixed directions $(l\,m\,n)\ (l'\,m'\,n')$ through a moving point J, then $\dfrac{JQ\,.\,JQ'}{JP\,.\,JP'} = \dfrac{\mathbf{A}l^2 + \mathbf{B}m^2 + \mathbf{C}n^2}{\mathbf{A}l'^2 + \mathbf{B}m'^2 + \mathbf{C}n'^2}$; so that the ratio of the rectangles made by the segments of two lines drawn through any point to meet a conicoid depends on their directions only. Page 185 is a particular case, when J is confined to a plane.

Conjugate Planes and Diameters.

Let $P_1 P_2$ be a chord of an ellipsoid in a given direction $(l\,m\,n)$. Let J be its middle point, so that the roots of equation (xxiv) are equal and opposite and the coefficient of r is zero. Thus the locus of J, the middle point of parallel chords, is $\mathbf{A}lx + \mathbf{B}my + \mathbf{C}nz = 0$, a plane through the centre.

Let OQ_1 be the diameter $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$, meeting the surface at $Q_1\ (x'y'z')$. Then the tangent plane at Q_1 is

$$\mathbf{A}x'x + \mathbf{B}y'y + \mathbf{C}z'z = 1,$$

and is parallel to the locus of J, since $\dfrac{x'}{l} = \dfrac{y'}{m} = \dfrac{z'}{n}$.

Hence the diametral plane $\mathbf{A}x'x + \mathbf{B}y'y + \mathbf{C}z'z = 0$ is the locus of J, and bisects all chords parallel to the diameter OQ_1 if the tangent plane at Q_1 is parallel to the diametral plane.

Let $Q_2\ (x''y''z'')$ be a point on this diametral plane and on the surface.

Then $\mathbf{A}x'x'' + \mathbf{B}y'y'' + \mathbf{C}z'z'' = 0.$

The symmetry of the equation shows that all chords parallel to OQ_2 are bisected by a plane containing OQ_1. (Compare pp. 159, 160.)

Now determine a point $Q_3(x'''y'''z''')$ on the surface, such that

$$\mathbf{A}x''x''' + \mathbf{B}y''y''' + \mathbf{C}z''z''' = 0,$$
$$\mathbf{A}x'''x' + \mathbf{B}y'''y' + \mathbf{C}z'''z' = 0.$$

Then, in whatever order we take the points, the diametral plane containing two of them is parallel to the tangent plane at the third, and bisects all chords parallel to the diameter through the third.

OQ_1, OQ_2, OQ_3 are called conjugate diameters of the ellipsoid, and the planes Q_2OQ_3, Q_3OQ_1, Q_1OQ_2 are conjugate planes. Q_1, Q_2, Q_3 are real, since the ellipsoid is a closed surface.

In other conicoids the locus of the middle points of parallel chords in direction $(l'm'n')$ is the plane $\mathbf{A}l'x + \mathbf{B}m'y + \mathbf{C}n'z = 0$.

Of the diametral planes in directions

$$(\mathbf{A}l'\ \mathbf{B}m'\ \mathbf{C}n')\ (\mathbf{A}l''\ \mathbf{B}m''\ \mathbf{C}n'')\ (\mathbf{A}l'''\ \mathbf{B}m'''\ \mathbf{C}n'''),$$

each bisects all chords parallel to the intersection of the other two if $\mathbf{A}l'l'' + \mathbf{B}m'm'' + \mathbf{C}n'n'' = \mathbf{A}l''l''' + \mathbf{B}m''m''' + \mathbf{C}n''n'''$

$$= \mathbf{A}l'''l' + \mathbf{B}m'''m' + \mathbf{C}n'''n' = 0.$$

The planes are then conjugate planes, and their lines of intersection *conjugate directions*, as in two dimensions.

Evidently OQ_1, OQ_2 are conjugate directions for the plane section Q_1OQ_2.

If we write $l_1 = \sqrt{\mathbf{A}}x'$, $l_2 = \sqrt{\mathbf{A}}x''$, ... $m_1 = \sqrt{\mathbf{B}}y'$..., the equations above, together with $\mathbf{A}x'^2 + \mathbf{B}y'^2 + \mathbf{C}z'^2 = 1$, &c., are equivalent to the six equations on p. 257. These are there shown to necessitate $1 = l_1^2 + l_2^2 + l_3^2 = $ &c.

$\therefore \dfrac{1}{\mathbf{A}} + \dfrac{1}{\mathbf{B}} + \dfrac{1}{\mathbf{C}} = x'^2 + x''^2 + x'''^2 + y'^2 + \ldots +$, and in the ellipsoid this becomes $\alpha^2 + \beta^2 + \gamma^2 = OQ_1^2 + OQ_2^2 + OQ_3^2$. (Compare p. 161.)

Using the method of p. 220, the *tangent plane* at $(x_1 y_1 z_1)$ on the surface $f(x\,y\,z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy - 1 = 0$ is $(x - x_1) \cdot D_x f + (y - y_1) \cdot D_y f + (z - z_1) \cdot D_z f = 0$, which reduces to $x(ax_1 + hy_1 + gz_1) + y(hx_1 + by_1 + fz_1) + z(gx_1 + fy_1 + cz_1)$

$$= f(x_1 y_1 z_1) = 1.$$

The *axes* of $f(x\,y\,z) = 0$, to which form the general equation can be reduced by transference to the centre as origin, can be obtained from the consideration that if OA is an axis it is perpendicular to the tangent plane at A.

Let $(l\,m\,n)$ be the direction OA, and $(x_1 y_1 z_1)$ the point A.

Then $\frac{x_1}{l} = \frac{y_1}{m} = \frac{z_1}{n}$.

We must then have

$$\frac{al + hm + gn}{l} = \frac{hl + bm + fn}{m} = \frac{gl + fm + cn}{n} = \lambda \text{ (say)}.$$

The elimination of l, m, n shows that λ is a root of the discriminating cubic.

Taking the root λ_1 (say) and solving for $l\,m\,n$ we have

$$l\,\{gh - (a - \lambda_1) f\} = m\,\{hf - (b - \lambda_1)\,g\} = n\,\{fg - (c - \lambda_1)\,h\}.$$

The three roots of the discriminating cubic then determine the directions of the axes, and we have determined l, m, n when $lx + my + nz = 0$ is a *principal plane.*

We have now included the principal analytical properties in three dimensions that have analogues or contain as special cases the results in pp. 130–86. There remain two properties of special importance inapplicable to two dimensions.

Generating Lines.

Returning to equation (xxiv), we find that *every* point on the line $(l\,m\,n)$ through $(x_1\,y_1\,z_1)$ lies on the conicoid, if

Absolute term: $\mathbf{A}x_1^2 + \mathbf{B}y_1^2 + \mathbf{C}z_1^2 = 1$. (Point on curve.)

Coefficient of r: $\mathbf{A}lx_1 + \mathbf{B}my_1 + \mathbf{C}nz_1 = 0$. (Line in tangent plane.)

Coefficient of r^2: $\mathbf{A}l^2 + \mathbf{B}m^2 + \mathbf{C}n^2 = 0$.

Eliminating n, we have

$$\mathbf{A}l^2(\mathbf{A}x_1^2 + \mathbf{C}z_1^2) + \mathbf{B}m^2(\mathbf{B}y_1^2 + \mathbf{C}z_1^2) + 2\mathbf{AB}lmx_1y_1 = 0.$$

The roots in $l \div m$ are real,

if $\mathbf{A}^2\mathbf{B}^2x_1^2y_1^2 \not< \mathbf{AB}(\mathbf{A}x_1^2 + \mathbf{C}z_1^2)(\mathbf{B}y_1^2 + \mathbf{C}z_1^2)$,

if $\mathbf{ABC}z_1^2(\mathbf{A}x_1^2 + \mathbf{B}y_1^2 + \mathbf{C}z_1^2) \not> 0$,

if $\mathbf{ABC} \not> 0$,

if one letter and only one is negative. [Three cannot be negative if the surface is real.]

Hence in the hyperboloid of one sheet two lines can be found through every point, which lie wholly on the curve. These are called *generating lines.*

If the same process is applied to the hyperbolic paraboloid, the same property is found; if to the cone, all lines through the vertex are generating lines; and if to the cylinder, the moving line which generates the surface is always a generating line.

Writing the hyperboloid in the form $\frac{x^2}{\alpha^2} - \frac{z^2}{\gamma^2} = 1 - \frac{y^2}{\beta^2}$, we readily see that all points on the line formed by the intersection of $\frac{x}{\alpha} - \frac{z}{\gamma} = \left(1 - \frac{y}{\beta}\right) k$ and $\left(\frac{x}{\alpha} + \frac{z}{\gamma}\right) k = 1 + \frac{y}{\beta}$ lie on the curve, whatever the value of k, and similarly with $\frac{x}{\alpha} + \frac{z}{\gamma} = \left(1 - \frac{y}{\beta}\right) k'$ and $\left(\frac{x}{\alpha} - \frac{z}{\gamma}\right) k' = 1 + \frac{y}{\beta}$.

For various values of k, k' these form two systems of lines; every line k intersects every line k', since

$$\frac{x}{\alpha(k+k')} = \frac{y}{\beta(kk'-1)} = \frac{z}{\gamma(k'-k)} = \frac{1}{kk'+1}$$

satisfies all the equations; while it is easily shown that no two lines of the same system intersect. The hyperboloid is in fact of the form of two waste-paper baskets placed bottom to bottom, each straw from left to right intersecting each from right to left.

It can be shown that if a line moves so as to intersect always three fixed non-intersecting lines, it generates a conicoid.

In the paraboloid $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = wz$, the systems may be written

$$\text{(i)}\quad \frac{x}{\alpha} - \frac{y}{\beta} = kw, \quad \frac{x}{\alpha} + \frac{y}{\beta} = \frac{z}{k},$$

$$\text{and (ii)}\quad \frac{x}{\alpha} + \frac{y}{\beta} = k'w, \quad \frac{x}{\alpha} - \frac{y}{\beta} = \frac{z}{k'},$$

and two lines intersect at $\frac{x}{\alpha(k+k')} = \frac{y}{\beta(k'-k)} = \frac{z}{2kk'} = \frac{w}{2}$.

The projections of the lines on the plane XOY are parallel to the asymptotes in that plane.

[The line EG drawn in Figure 97 is the projection of the generating line $x - \sqrt{14} = 2y + \sqrt{14} = z\sqrt{\frac{2}{7}}$.

The length EG is $\sqrt{70}$, and G is 14 above E. The gradient is therefore $14 \div \sqrt{70} = \tan 59°8'$.]

Circular Sections.

In connexion with equation (xxiv) let J be the centre of the conicoid, and let JPP' be confined to the diametral plane $(l_1\, m_1\, n_1)$. Required to determine the plane $(l_1\, m_1\, n_1)$ so that the section on it shall be circular.

We must have that $ll_1 + mm_1 + nn_1 = 0$, and that the roots of (xxiv), which then becomes $r^2(\mathbf{A}l^2 + \mathbf{B}m^2 + \mathbf{C}n^2) - 1 = 0$, must be independent of $l\, m\, n$.

Let $\mathbf{A}$ be algebraically the greatest and $\mathbf{C}$ the least of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$.

$\mathbf{A}l^2 + \mathbf{B}m^2 + \mathbf{C}n^2 = (\mathbf{A}-\mathbf{B})\,l^2 - (\mathbf{B}-\mathbf{C})\,n^2 + \mathbf{B}$ from equation (i).

Choose $l_1\, m_1\, n_1$ so that $l_1^2 : n_1^2 = \mathbf{A}-\mathbf{B} : \mathbf{B}-\mathbf{C}$, and $m_1 = 0$.

Then $ll_1 + nn_1 = 0$, and $l^2 : n^2 = n_1^2 : l_1^2$.

$\therefore\ (\mathbf{A}-\mathbf{B})\,l^2 = (\mathbf{B}-\mathbf{C})\,n^2$.

Hence $\mathbf{A}l^2 + \mathbf{B}m^2 + \mathbf{C}n^2 = \mathbf{B}$, and $r = \pm\dfrac{1}{\sqrt{\mathbf{B}}}$, for all directions in the two diametral planes $\dfrac{x}{\sqrt{\mathbf{A}-\mathbf{B}}} = \pm\dfrac{z}{\sqrt{\mathbf{B}-\mathbf{C}}}$, which are therefore circular sections.

Since parallel planes cut off similar sections, all sections in these directions are circular.

Of course, if $\mathbf{A}$ or $\mathbf{C}$ is the intermediate quantity, the equation is modified.

This analysis applies to all central conicoids, and can be adapted to the elliptic paraboloid, to the cone, and to the cylinder $\mathbf{A}x^2 + \mathbf{B}y^2 = k$, if $\mathbf{A}$, $\mathbf{B}$, k are positive.

In surfaces of revolution the directions become coincident.

The points of contact of tangent planes parallel to circular sections are called *umbilics*; in the neighbourhood of such points the surface approximates to one of revolution.

Example on pp. 262–5. Show that the area of the section of

$$\mathbf{A}x^2 + \mathbf{B}y^2 + \mathbf{C}z^2 = 1 \text{ by } lx + my + nz = p$$

is

$$\pi\,(p_1^2 - p^2) \div p_1^3 . \sqrt{\mathbf{ABC}},$$

where $p_1^2 = l^2/\mathbf{A} + m^2/\mathbf{B} + n^2/\mathbf{C}$. [Project on plane $z = 0$.]

NOTE ON THE WORDS 'IRRATIONAL' AND 'INCOMMENSURABLE'

IN this book we have followed the usage which has been common of keeping the word *irrational* for surds, that is roots of numbers whose roots cannot be expressed as an integer (n) or the ratio of two integers ($p : q$); while the word *incommensurable* has been applied for all other quantities, such as e, π, logarithms, trigonometrical ratios, &c., whether algebraic or geometric, which are not of the form 0, n, or $p \div q$.

But in modern mathematics it is desired to base the whole of algebra on a system of definitions and rules which are entirely independent of space or physical quantity, and since the science is then independent of measurements the word 'incommensurable' is inappropriate. Also it is contrary to the principle of economy in the use of terms to use three words where only two ideas are to be expressed.

It has been therefore authoritatively suggested that all algebraic numbers, whether resulting from obtaining the limit of infinite series or any analytic process,* which cannot be expressed as 0, n, or $p \div q$, shall be termed *irrational*, including the species *surds*; while two geometric or two physical quantities, which are such that the ratio of their algebraic measurements is irrational, shall be termed *incommensurable* with each other. The word incommensurable will then not occur in pure algebra.

* A surd can be expressed as an infinite series thus:

$$\sqrt{7} = \sqrt{9-2} = 3\left(1-\tfrac{2}{9}\right)^{\frac{1}{2}} = 3\left(1-\tfrac{1}{2}\cdot\tfrac{2}{9}-\tfrac{1}{8}\cdot\tfrac{4}{81}-\right),$$

and in other ways.

ANSWERS TO EXAMPLES

PAGE 7.

2. $3^{\frac{11}{4}} \times 2^{\frac{3}{2}}$. **3.** $3\sqrt{6}$. **4.** $1 \div \sqrt[6]{b^5}$. **5.** $a-b$. **6.** $a^{\frac{2}{3}} \mp a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$.

PAGE 10.

1. The logs of 120, 125, 128, 121 are 2·079, 2·097, 2·107, 2·083.
2. The logs of 13, 17, 19 are 1·114, 1·230, 1·279.
4. 1·26, 1·183, ·431. **5.** $17\frac{1}{2}$ years; £357. **6.** 2·47. **7.** 1016.

PAGE 13.

2. $x > 1$ or < -4. **3.** $-b > a$.

PAGE 16.

1. 143. **2.** 2046. **3.** ·499992. **4.** £1359.
5. $\frac{1}{30}(3n^2+3n-1)(2n+1)(n+1)n$. **6.** $\frac{1}{3}n(n+1)(n+2)$.
7. $\frac{1}{4}n(n+1)(n+2)(n+3)$. **8.** $\frac{1}{6}n(n+1)(n+2)$.

PAGE 24.

1. 64. **2.** 40320; 4. **3.** 1·06152; $1{\cdot}2668 \times 10^{16}$.
4. $x^6 - 6x^4 + 15x^2 - 20 + 15x^{-2} - 6x^{-4} + x^{-6}$.
5. ${}_{12}C_t x^{24-3t}.2^t$; $112640\,x^{-3}$.

PAGE 55.

4. (i) $n180° + (-1)^n 30°$. (ii) $n180° \pm 22\frac{1}{2}°$. (iv) $n180° + 63\frac{1}{2}°$.
5. (i) 90°, 26°, 64°; 333. (ii) 32·8, 133°, 20°; 301.
(iii) 27°, 17·6, 21·7; 86·6. (iv) 121°, 34°, 6·1; 52 or 9°, 146°, 1·1; ·93.

PAGE 63.

1, 2, and 3.

	3°	33°	$7\frac{1}{2}°$.
sin	·052	·545	·131.
cos	·999	·839	·991.
tan	·052	·649	·132.

4. $n360° - 37°$.
7. 35°, 86°, 59°; 12.
8. 104°, 51°.

PAGE 85.

1. $y = x^{\frac{1\cdot5}{2}}$; $p = q\cos q - \sin q\sqrt{1-q^2}$.

PAGE 97.

2. $-8{\cdot}58257$, ·58257. **3.** 2, 4, -1.
6. $x^4 - 16x^3 + 85x^2 - 158x + 46 = 0$. **7.** 2·0927.
8. 2·2338, 2·7185, $-3{\cdot}9522$.

Page 105.

The limits are (*a*) 2, (*b*) $-\frac{4}{3}$, (*c*) 12, (*d*) $-\frac{5}{2}, \frac{3}{4}$, (*e*) 1, (*f*) 0, (*g*) 0.

Page 106.

2. (i) $n(n+1)$. (ii) $2n(n+1)^2 x^{n-1}$.

Page 118.

4. Error is less than ·16 of $\frac{a}{b}$.

If m, n, and d_2 are positive, the error is $-mnd_1d_2 + R$, where

$$R < \tfrac{1}{2} m(m-1) \cdot \frac{d_1^2}{1-d_1}(1 - nd_2 + \tfrac{1}{2} n\,\overline{n+1}\, d_2^2) + \tfrac{1}{2} md_1d_2n(n+1).$$

Page 137.

1. $8x - 9y = 12$. **2.** $5x + 4y = 2$, $4x - 5y + 23 = 0$.

3. $x - y + 1 = 0$, $x - 3y = 11$, $x - 5y = 23$; $(-7, -6)$.

4. $11x + 3y = 17$, $x = 2$, $7x + 3y = 9$; $(2, -\frac{5}{3})$.

Page 141.

2. $33(y-2) = (x-3)(30 \pm 17\sqrt{3})$.

4. $x \pm y = 0$, $8x + 4y = 5$, $-2x + 4y = 5$, $3x + 9y = 5$, $3x - y = 5$.
Inscribed centre $(\frac{5}{12}, \frac{5}{12})$; escribed centres $(\frac{5}{2}, \frac{5}{2})$, $(\frac{5}{4}, -\frac{5}{4})$, $(-\frac{5}{6}, \frac{5}{6})$.

5. A straight line.

Page 148.

Equation may be written $\dfrac{(x-2)^2}{4^2} - \dfrac{(y+4)^2}{3^2} = 1$.

Page 150.

	Centre.	Vertices (tranverse axis).	Asymptotes.	Semi-Axes: lengths.	Equations.
1.	$-2, 0$	...	...	2	...
2.	$0, 0$	$\pm 5, 0$	...	5, 3	$y = 0$, $x = 0$.
3.	$0, 0$	$\pm 5, 0$	$3x \pm 5y = 0$	5	$y = 0$, $x = 0$.
4.	$0, 0$	$\pm 3, 0$	$5x \pm 3y = 0$	3	$y = 0$, $x = 0$.
5.	$-1, 5$	$-1, 5 \pm 5{\cdot}4$	...	5·4, 2·7	$x = -1$, $y = 5$.
6.	$-\frac{15}{16}, 0$	$-\frac{15}{16}, \pm 1{\cdot}2$	$x + \frac{15}{16} \pm \sqrt{2}\,y = 0$	1·2	$16x + 15 = 0$, $y = 0$.
7.	$0, -\frac{3}{4}$	$\pm 2{\cdot}3, 0$	...	2·3, 1·6	$y = -\frac{3}{4}$, $x = 0$.

Page 182.

	Centre.	Semi-Axes: length.	Equations.
1. (i) Ellipse	$-5{\cdot}5, -1{\cdot}3$	5·66, 2·04	$y = {\cdot}27x + {\cdot}2$, $x + {\cdot}27y + 5{\cdot}9 = 0$.
(ii) Hyperbola	$-{\cdot}68, -{\cdot}37$	·9	$y = {\cdot}56x + {\cdot}02$, $x + {\cdot}56y + {\cdot}88 = 0$
(iii) Hyperbola	$0, 1\frac{1}{7}$	1·32	$y = 1{\cdot}51x + 1{\cdot}1$, $x + 1{\cdot}51y = 1{\cdot}73$.

(iv) Parabola. Vertex ·62, −·08. Axis $2x + 3y = 1$.

(v) The straight lines $3x + 4y = 5$, $2x - 3y + 2 = 0$.

Page 200.

1. (i) $-2\sin 2x$. (ii) $2\sec^2 2x$. (iii) $2\cos x - 3\sin x$. (iv) $4x-3$.
(v) $9x^2-4$. (vi) $-\dfrac{1}{x^2}$. (vii) $-2\div(x+3)^2$. (viii) $-\frac{2}{9}x\,(1-\frac{1}{9}x^2)^{-\frac{3}{2}}$.

Page 206.

3. Max. when $x = 2 - \dfrac{1}{\sqrt{3}}$, min. when $x = 2 + \dfrac{1}{\sqrt{3}}$.
4. Min. when $x = -1{\cdot}85$. (Use Horner's method, p. 95.)
5. Max. when $x = n\,.\,360^\circ + 33^\circ\,41'$, min. when $x = n\,.\,360^\circ + 213^\circ\,41'$.
6. Min. when $x = \dfrac{1}{e}$. 7. Min. when $x = \frac{1}{3}$, max. when $x = 0$.

Page 216.

1. $1\frac{5}{6}$. 2. $1 - \dfrac{1}{\sqrt{2}}$. 3. $9\log_{10}e$. 4. $\log_e a$.

Page 217.

1. $u-gt$, $ut - \frac{1}{2}gt^2$. 2. $\sqrt{\dfrac{g}{k}}$.

3. (i) $y = \frac{1}{2}ax^2 + bx + C$. (ii) $y = \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + cx + C$.
(iii) $y = C - a\cos c$. (iv) $2y = e^x(\sin x - \cos x) + C$.
(v) $y^2 = 4ax + C$. (vi) $3(x+y)^2 = 2x^3 + C$.
(vii) $\tan y + \cos x = C$. (viii) $y = \pm a\sin^{-1}x + C$.
(ix) $y\,.\cos x = C$. (x) $8y = ax^4 + C_1x^2 + C_2$

Page 232.

Modulus 2, amplitudes 0, $\pm\dfrac{2\pi}{7}$, $\pm\dfrac{4\pi}{7}$, $\pm\dfrac{6\pi}{7}$.
$\pm 3{\cdot}56$ and $1{\cdot}78\,(\pm 1 \pm \iota\,.\,1{\cdot}732)$.

Page 249.

$\sinh 1{\cdot}44\ldots = 2$; $\tanh {\cdot}423\ldots = {\cdot}4$.

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